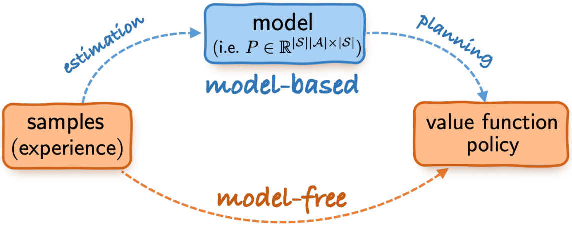
Reinforcement learning (Part 2): Model-free RL

Yuxin Chen

Wharton Statistics & Data Science, Spring 2022

Model-based vs. model-free RL



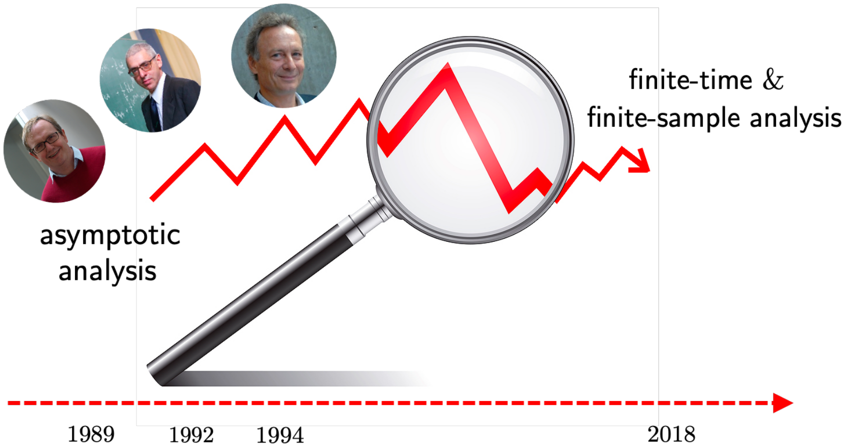
Model-based approach ("plug-in")

1. build empirical estimate $\widehat{P}$ for P
2. planning based on empirical $\widehat{P}$

Model-free approach

— learning w/o modeling & estimating environment explicitly

— memory-efficient, online, ...



Focus of this part: classical **Q-learning** algorithm and beyond

Model-free RL

1. Basics of Q-learning
2. Synchronous Q-learning and variance reduction (simulator)
3. Asynchronous Q-learning (Markovian data)
4. Q-learning with lower confidence bounds (offline RL)
5. Q-learning with upper confidence bounds (online RL)

A starting point: Bellman optimality principle

Bellman operator

$$\mathcal{T}(Q)(s,a) := \underbrace{r(s,a)}_{\text{immediate reward}} + \gamma \mathop{\mathbb{E}}_{s' \sim P(\cdot|s,a)} \underbrace{\left[\max_{a' \in \mathcal{A}} Q(s',a')\right]}_{\text{next state's value}}$$

- one-step look-ahead

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Bellman equation: $Q^\star$ is *unique* solution to

$$\mathcal{T}(Q^\star) = Q^\star$$

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Bellman equation: $Q^\star$ is *unique* solution to

$$\mathcal{T}(Q^\star) = Q^\star$$

- **takeaway message:** it suffices to solve the Bellman equation
- **challenge:** how to solve it using stochastic samples?

Richard Bellman

A detour: stochastic approximation

- **Goal:** solve
$$G(x) = \mathbb{E}[g(x;\xi)] = 0$$
  - ξ: randomness in problem

- **What we can query:** for any given input x, we receive a *random* sample $g(x;\xi)$ obeying $\mathbb{E}[g(x;\xi)] = G(x)$

Stochastic approximation (Robbins, Monro '51)


Herbert Robbins


Sutton Monro

stochastic approximation

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^{t} - \eta_t \, \boldsymbol{g}(\boldsymbol{x}^t; \boldsymbol{\xi}^t) \tag{1}$$

where $\boldsymbol{g}(\boldsymbol{x}^t; \boldsymbol{\xi}^t)$ is *unbiased* estimate of $\boldsymbol{G}(\boldsymbol{x}^t)$, i.e.

$$\mathbb{E}\big[\boldsymbol{g}(\boldsymbol{x}^t; \boldsymbol{\xi}^t)\big] = \boldsymbol{G}(\boldsymbol{x}^t)$$

Stochastic approximation (Robbins, Monro '51)

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stochastic approximation

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \eta_t \, \boldsymbol{g}(\boldsymbol{x}^t; \boldsymbol{\xi}^t) \tag{1}$$

a stochastic algorithm for finding roots of $\boldsymbol{G}(\boldsymbol{x}) := \mathbb{E}[\boldsymbol{g}(\boldsymbol{x}; \boldsymbol{\xi})]$

Q-learning: a stochastic approximation algorithm

Chris Watkins

Peter Dayan

$\underbrace{\text{Stochastic approximation}}_{\text{Robbins \& Monro, 1951}}$ for solving the **Bellman equation**

$$\mathcal{T}(Q) - Q = 0$$

where

$$\mathcal{T}(Q)(s,a) := \underbrace{r(s,a)}_{\text{immediate reward}} + \gamma \mathop{\mathbb{E}}_{s' \sim P(\cdot|s,a)} \underbrace{\left[\max_{a' \in \mathcal{A}} Q(s',a') \right]}_{\text{next state's value}}.$$

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Stochastic approximation for solving Bellman equation $\mathcal{T}(Q) - Q = 0$

$$\underbrace{Q_{t+1}(s,a) = Q_t(s,a) + \eta_t\big(\mathcal{T}_t(Q_t)(s,a) - Q_t(s,a)\big)}_{\text{sample transition } (s,a,s')}, \quad t \geq 0$$

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Stochastic approximation for solving Bellman equation $\mathcal{T}(Q) - Q = 0$

$$\underbrace{Q_{t+1}(s,a) = (1-\eta_t)Q_t(s,a) + \eta_t \mathcal{T}_t(Q_t)(s,a)}_{\text{sample transition } (s,a,s')}, \quad t \geq 0$$

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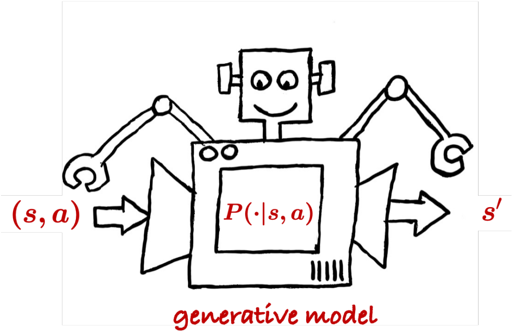
A generative model / simulator

— *Kearns, Singh '99*



In each iteration, collect an independent sample (s, a, s') for each (s, a)

Synchronous Q-learning

Chris Watkins

Peter Dayan

for $t = 0, 1, \ldots, T$

for each $(s, a) \in \mathcal{S} \times \mathcal{A}$

draw a sample (s, a, s'), run

$$Q_{t+1}(s, a) = (1 - \eta_t) Q_t(s, a) + \eta_t \Big\{ r(s, a) + \gamma \max_{a'} Q_t(s', a') \Big\}$$

synchronous: all state-action pairs are updated simultaneously

Sample complexity of synchronous Q-learning

Theorem 1 (Li, Cai, Chen, Gu, Wei, Chi '21)

For any $0 < \varepsilon \le 1$, *synchronous Q-learning yields* $\|\widehat{Q} - Q^\star\|_\infty \le \varepsilon$ *with high prob., with sample complexity (i.e.,* $T|\mathcal{S}||\mathcal{A}|$*) at most*

$$\widetilde{O}\left(\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^4\varepsilon^2}\right)$$

other papers	sample complexity
Even-Dar & Mansour '03	$2^{\frac{1}{1-\gamma}} \frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^4\varepsilon^2}$
Beck & Srikant '12	$\frac{\lvert\mathcal{S}\rvert^2\lvert\mathcal{A}\rvert^2}{(1-\gamma)^5\varepsilon^2}$
Wainwright '19	$\frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^5\varepsilon^2}$
Chen et al. '20	$\frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^5\varepsilon^2}$

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Theorem 1 (Li, Cai, Chen, Gu, Wei, Chi '21)

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$$\widetilde{O}\left(\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^4\varepsilon^2}\right)$$

- Covers both *constant* and *rescaled linear* learning rates:

$$\eta_t \equiv \frac{1}{1+\frac{c_1(1-\gamma)T}{\log^2 T}}$$

or $$\eta_t = \frac{1}{1+\frac{c_2(1-\gamma)t}{\log^2 T}}$$

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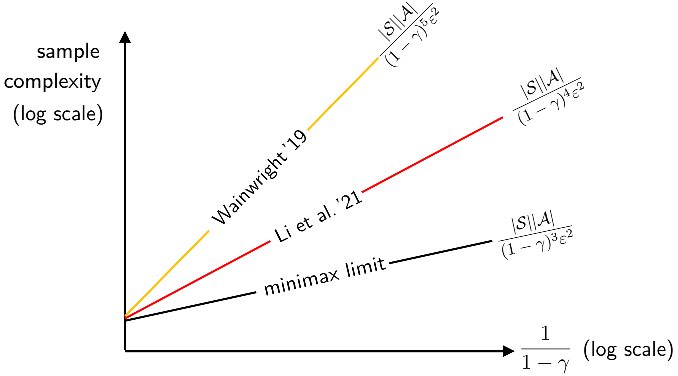
All this requires sample size at least $\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^4\varepsilon^2}$ ...



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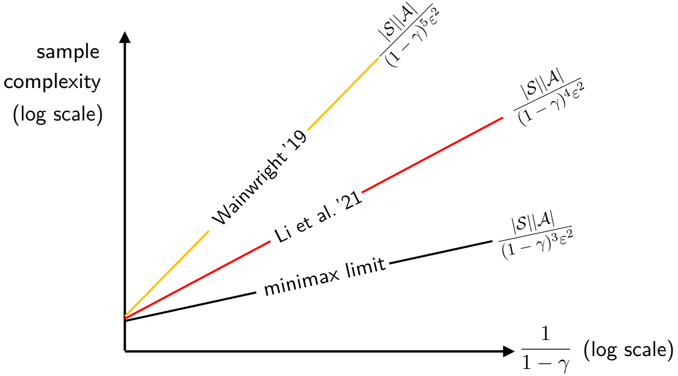
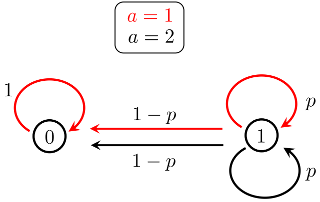


Question: *Is Q-learning sub-optimal, or is it an analysis artifact?*

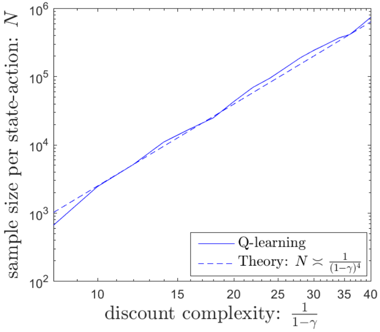
A numerical example: $\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^4\varepsilon^2}$ samples seem necessary ...

— *observed in Wainwright '19*



$$p = \frac{4\gamma - 1}{3\gamma}$$

$$r(0,1) = 0, \quad r(1,1) = r(1,2) = 1$$



Q-learning is NOT minimax optimal

Theorem 2 (Li, Cai, Chen, Gu, Wei, Chi, 2021)

For any $0 < \varepsilon \leq 1$, there exist an MDP such that to achieve $\|\widehat{Q} - Q^{\star}\|_{\infty} \leq \varepsilon$, synchronous Q-learning needs at least

$$\widetilde{\Omega}\left(\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^4\varepsilon^2}\right) \quad \textit{samples}$$

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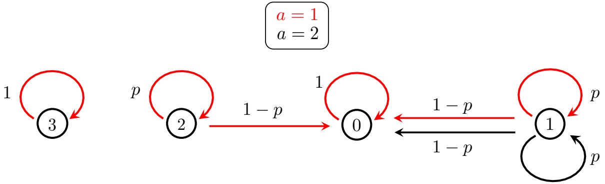
- Tight **algorithm-dependent** lower bound
- Holds for both constant and rescaled linear learning rates



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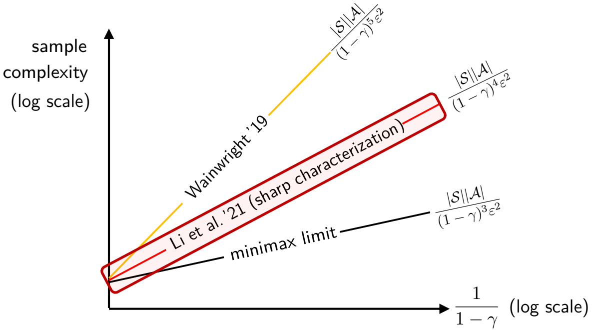

Why is Q-learning sub-optimal?

Over-estimation of Q-functions (Thrun & Schwartz '93; Hasselt '10)

- $\max_{a \in \mathcal{A}} \mathbb{E}[X(a)]$ tends to be over-estimated (high positive bias) when $\mathbb{E}[X(a)]$ is replaced by its empirical estimates using a small sample size
- often gets worse with a large number of actions (Hasselt, Guez, Silver '15)

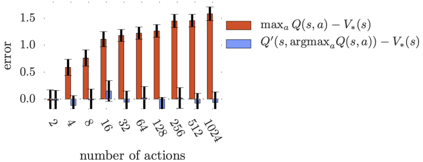

Figure 1: The orange bars show the bias in a single Q-learning update when the action values are $Q(s,a) = V_*(s) + \epsilon_a$ and the errors $\{\epsilon_a\}_{a=1}^m$ are independent standard normal random variables. The second set of action values Q', used for the blue bars, was generated identically and independently. All bars are the average of 100 repetitions.

Improving sample complexity via variance reduction

A detour: finite-sum optimization

$$\text{minimize}_{\boldsymbol{x}\in\mathbb{R}^d} \qquad F(\boldsymbol{x}) := \frac{1}{n}\sum_{i=1}^{n} f_i(\boldsymbol{x})$$

- $F(\cdot)$: μ-strongly convex
- f_i: convex and L-smooth (i.e., ∇f_i is L-Lipschitz)
- $\kappa := L/\mu$: condition number

Recall: SGD theory with fixed stepsizes

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \eta_t \, \boldsymbol{g}^t$$

- $\boldsymbol{g}^t$: an unbiased stochastic estimate of $F(\boldsymbol{x}^t)$
- $\mathbb{E}[\|\boldsymbol{g}^t\|_2^2] \leq \sigma_{\mathrm{g}}^2 + c_{\mathrm{g}}\|\nabla F(\boldsymbol{x}^t)\|_2^2$

This SGD-type algorithm with $\eta_t \equiv \eta$ obeys

$$\mathbb{E}[F(\boldsymbol{x}^t) - F(\boldsymbol{x}^*)] \leq \frac{\eta L \sigma_{\mathrm{g}}^2}{2\mu} + (1-\eta\mu)^t \big(F(\boldsymbol{x}^0) - F(\boldsymbol{x}^*)\big)$$

Recall: SGD theory with fixed stepsizes

$$\mathbb{E}[F(\boldsymbol{x}^t) - F(\boldsymbol{x}^*)] \leq \frac{\eta L \sigma_{\rm g}^2}{2\mu} + (1-\eta\mu)^t \big(F(\boldsymbol{x}^0) - F(\boldsymbol{x}^*)\big)$$

- vanilla SGD: $\boldsymbol{g}^t = \nabla f_{i_t}(\boldsymbol{x}^t)$
  - **issue:** $\sigma_{\rm g}^2$ is non-negligible even when $\boldsymbol{x}^t = \boldsymbol{x}^*$
- **question:** it is possible to design $\boldsymbol{g}^t$ with reduced variability $\sigma_{\rm g}^2$?

A simple idea

Imagine we take some $\boldsymbol{v}^t$ with $\mathbb{E}[\boldsymbol{v}^t] = \mathbf{0}$ and set

$$\boldsymbol{g}^t = \nabla f_{i_t}(\boldsymbol{x}^t) - \boldsymbol{v}^t$$

— so $\boldsymbol{g}^t$ is still an unbiased estimate of $\nabla F(\boldsymbol{x}^t)$

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question: how to reduce variability (i.e. $\mathbb{E}[\|\boldsymbol{g}^t\|_2^2] < \mathbb{E}[\|\nabla f_{i_t}(\boldsymbol{x}^t)\|_2^2]$)?

answer: find some zero-mean $\boldsymbol{v}^t$ that is positively correlated with $\nabla f_{i_t}(\boldsymbol{x}^t)$ (i.e. $\langle \boldsymbol{v}^t, \nabla f_{i_t}(\boldsymbol{x}^t)\rangle > 0$) (why?)

Reducing variance via gradient aggregation

If the current iterate is not too far away from previous iterates, then historical gradient info might be useful in producing such a $\boldsymbol{v}^t$ to reduce variance

main idea of variance reduction: aggregate previous gradient info to help improve the convergence rate

Stochastic variance reduced gradient (SVRG)

— *Johnson, Zhang '13*

key idea: if we have access to a history point $\boldsymbol{x}^{\text{old}}$ and $\nabla F(\boldsymbol{x}^{\text{old}})$, then

$$\underbrace{\nabla f_{i_t}(\boldsymbol{x}^t) - \nabla f_{i_t}(\boldsymbol{x}^{\text{old}})}_{\to \mathbf{0} \text{ if } \boldsymbol{x}^t \approx \boldsymbol{x}^{\text{old}}} + \underbrace{\nabla F(\boldsymbol{x}^{\text{old}})}_{\to \mathbf{0} \text{ if } \boldsymbol{x}^{\text{old}} \approx \boldsymbol{x}^*} \qquad \text{with } i_t \sim \text{Unif}(1, \cdots, n)$$

- is an unbiased estimate of $\nabla F(\boldsymbol{x}^t)$
- $\underbrace{\text{converges to } \mathbf{0}}_{\text{variability is reduced!}}$ if $\boldsymbol{x}^t \approx \boldsymbol{x}^{\text{old}} \approx \boldsymbol{x}^*$

Stochastic variance reduced gradient (SVRG)

- operate in epochs
- in the s^{th} epoch
  - **very beginning**: take a snapshot $\boldsymbol{x}_s^{\text{old}}$ of the current iterate, and compute the *batch* gradient $\nabla F(\boldsymbol{x}_s^{\text{old}})$
  - **inner loop**: use the snapshot point to help reduce variance

$$\boldsymbol{x}_s^{t+1} = \boldsymbol{x}_s^t - \eta\big\{\nabla f_{i_t}(\boldsymbol{x}_s^t) - \nabla f_{i_t}(\boldsymbol{x}_s^{\text{old}}) + \nabla F(\boldsymbol{x}_s^{\text{old}})\big\}$$

a hybrid approach: batch gradient is computed only once per epoch

Remark

- constant stepsize η
- each epoch contains $2m + n$ gradient computations
  - the batch gradient is computed only once every m iterations
  - the average per-iteration cost of SVRG is comparable to that of SGD if $m \gtrsim n$
- linear convergence

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  - the batch gradient is computed only once every m iterations
  - the average per-iteration cost of SVRG is comparable to that of SGD if $m \gtrsim n$
- linear convergence
- **total computational cost:**

$$\underbrace{(m+n)}_{\text{number of grad computation per epoch}} \log\frac{1}{\varepsilon} \asymp \underbrace{(n+\kappa)\log\frac{1}{\varepsilon}}_{\text{if } m \asymp \max\{n,\kappa\}}$$

Back to Q-learning . . .

— *inspired by Johnson & Zhang '13*

Variance-reduced Q-learning updates (Wainwright '19)

$$Q_t(s,a) = (1-\eta)Q_{t-1}(s,a) + \eta\Big(\mathcal{T}_t(Q_{t-1}) \underbrace{-\mathcal{T}_t(\overline{Q}) + \widetilde{\mathcal{T}}(\overline{Q})}_{\text{use } \overline{Q} \text{ to help reduce variability}}\Big)(s,a)$$

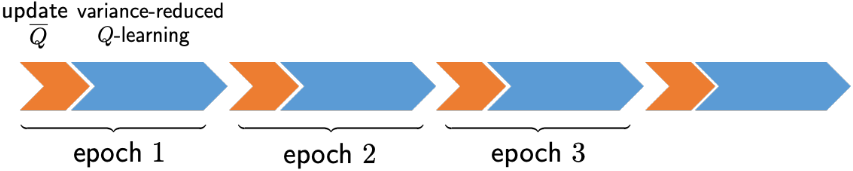
- $\overline{Q}$: some reference Q-estimate
- $\widetilde{\mathcal{T}}$: empirical Bellman operator (using a *batch* of samples)

$$\mathcal{T}_t(Q)(s,a) = r(s,a) + \gamma \max_{a'} Q(s',a')$$

$$\widetilde{\mathcal{T}}(Q)(s,a) = r(s,a) + \gamma \mathop{\mathbb{E}}_{s' \sim \widetilde{P}(\cdot|s,a)} \Big[\max_{a'} Q(s',a')\Big]$$

An epoch-based stochastic algorithm

— *inspired by Johnson & Zhang '13*



for each epoch

1. update $\overline{Q}$ and $\widetilde{\mathcal{T}}(\overline{Q})$ (which stay fixed in the rest of the epoch)
2. run variance-reduced Q-learning updates iteratively

Sample complexity of variance-reduced Q-learning

Theorem 3 (Wainwright '19)

For any $0 < \varepsilon \le 1$, sample complexity for ***variance-reduced synchronous Q-learning*** *to yield $\|\widehat{Q} - Q^\star\|_\infty \le \varepsilon$ is at most*

$$\widetilde{O}\Big(\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^3\varepsilon^2}\Big)$$

- allows for more aggressive learning rates

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$$\widetilde{O}\left(\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^3\varepsilon^2}\right)$$

- allows for more aggressive learning rates
- minimax-optimal for $0 < \varepsilon \leq 1$
  - remains suboptimal if $1 < \varepsilon < \frac{1}{1-\gamma}$

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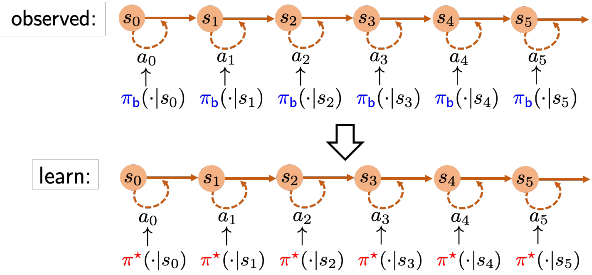
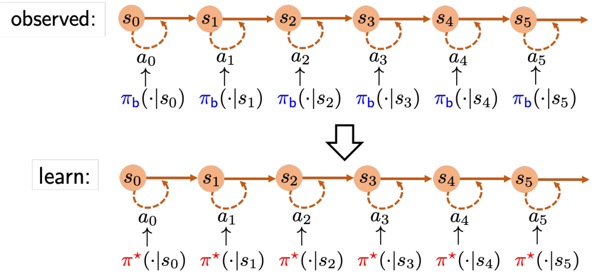
Markovian samples and behavior policy



Observed: $\underbrace{\{s_t, a_t, r_t\}_{t\geq 0}}_{\text{stationary Markovian trajectory}}$ generated by behavior policy π_b

Goal: learn optimal value $V^\star$ and $Q^\star$ based on sample trajectory

Markovian samples and behavior policy



Key quantities of sample trajectory

- minimum state-action occupancy probability (uniform coverage)

$$\mu_{\mathsf{min}} := \min \underbrace{\mu_{\pi_{\mathsf{b}}}(s,a)}_{\text{stationary distribution}}$$

- mixing time: t_{mix}

Q-learning on Markovian samples

Chris Watkins

Peter Dayan

$$\underbrace{Q_{t+1}(s_t, a_t) = (1 - \eta_t)Q_t(s_t, a_t) + \eta_t \mathcal{T}_t(Q_t)(s_t, a_t)}_{\textit{only} \text{ update } (s_t, a_t)\text{-th entry}}, \quad t \geq 0$$

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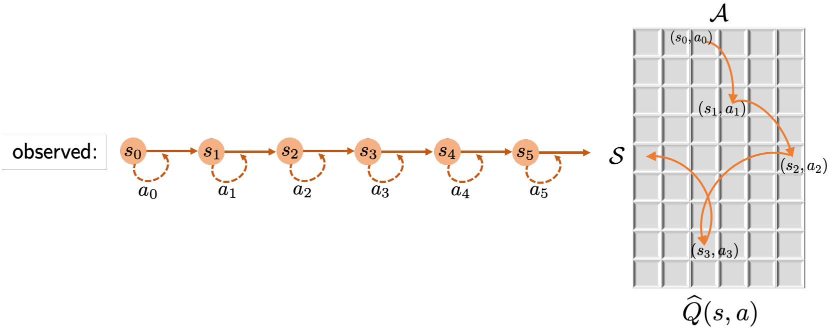
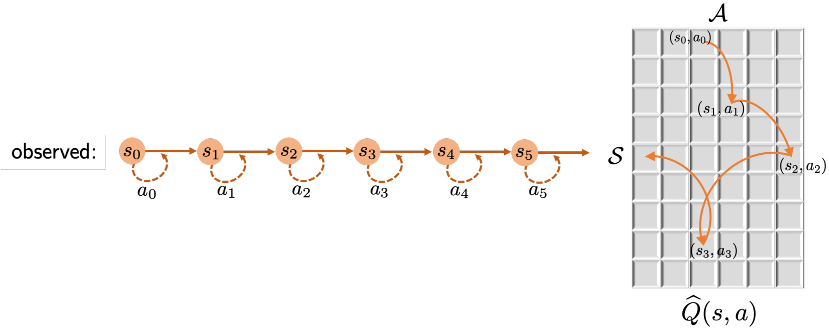
$$\mathcal{T}_t(Q)(s_t, a_t) = r(s_t, a_t) + \gamma \max_{a'} Q(s_{t+1}, a')$$

Q-learning on Markovian samples



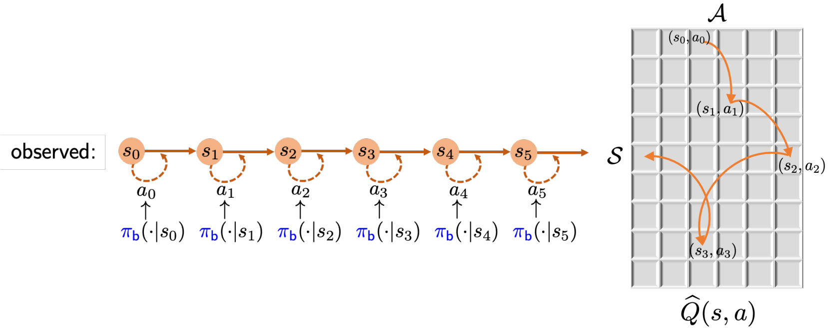
- **asynchronous:** only a single entry is updated each iteration

Q-learning on Markovian samples



- **asynchronous:** only a single entry is updated each iteration
  - resembles Markov-chain *coordinate descent*

Q-learning on Markovian samples



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  - resembles Markov-chain *coordinate descent*

- **off-policy:** target policy $\pi^\star \neq$ behavior policy π_b

A highly incomplete list of works

- Watkins, Dayan '92
- Tsitsiklis '94
- Jaakkola, Jordan, Singh '94
- Szepesvári '98
- Borkar, Meyn '00
- Even-Dar, Mansour '03
- Beck, Srikant '12
- Chi, Zhu, Bubeck, Jordan '18
- Lee, He '18
- Chen, Zhang, Doan, Maguluri, Clarke '19
- Du, Lee, Mahajan, Wang '20
- Chen, Maguluri, Shakkottai, Shanmugam '20
- Qu, Wierman '20
- Devraj, Meyn '20
- Weng, Gupta, He, Ying, Srikant '20
- Li, Wei, Chi, Gu, Chen '20
- Li, Cai, Chen, Gu, Wei, Chi '21
- Chen, Maguluri, Shakkottai, Shanmugam '21
- ...

Sample complexity of asynchronous Q-learning

Theorem 4 (Li, Cai, Chen, Gu, Wei, Chi '21)

For any $0 < \varepsilon \leq \frac{1}{1-\gamma}$*, sample complexity of async Q-learning to yield* $\|\widehat{Q} - Q^{\star}\|_{\infty} \leq \varepsilon$ *is at most (up to log factor)*

$$\frac{1}{\mu_{\mathsf{min}}(1-\gamma)^4\varepsilon^2} + \frac{t_{\mathsf{mix}}}{\mu_{\mathsf{min}}(1-\gamma)}$$

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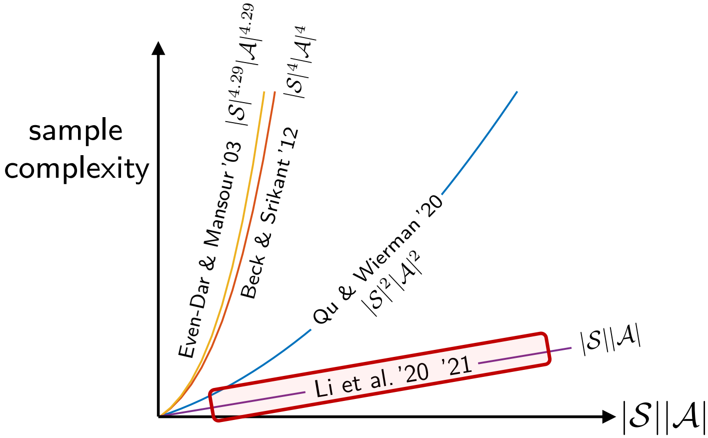
- **learning rates:** constant & rescaled linear

other papers	sample complexity
Even-Dar et al. '03	$\frac{(t_{\mathsf{cover}})^{\frac{1}{1-\gamma}}}{(1-\gamma)^4\varepsilon^2}$
Even-Dar et al. '03	$\left(\frac{t_{\mathsf{cover}}^{1+3\omega}}{(1-\gamma)^4\varepsilon^2}\right)^{\frac{1}{\omega}} + \left(\frac{t_{\mathsf{cover}}}{1-\gamma}\right)^{\frac{1}{1-\omega}}, \ \omega \in (\frac{1}{2}, 1)$
Beck & Srikant '12	$\frac{t_{\mathsf{cover}}^3 \lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^5\varepsilon^2}$
Qu & Wierman '20	$\frac{t_{\mathsf{mix}}}{\mu_{\min}^2(1-\gamma)^5\varepsilon^2}$
Li et al. '20	$\frac{1}{\mu_{\min}(1-\gamma)^5\varepsilon^2} + \frac{t_{\mathsf{mix}}}{\mu_{\min}(1-\gamma)}$
Chen et al. '21	$\frac{1}{\mu_{\min}^3(1-\gamma)^5\varepsilon^2} + \text{other-term}(t_{\mathsf{mix}})$

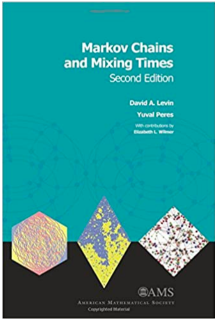
Linear dependency on $1/\mu_{\text{min}}$



if we take $\mu_{\text{min}} \asymp \frac{1}{|\mathcal{S}||\mathcal{A}|}$, $t_{\text{cover}} \asymp \frac{t_{\text{mix}}}{\mu_{\text{min}}}$

Effect of mixing time on sample complexity

$$\frac{1}{\mu_{\min}(1-\gamma)^4\varepsilon^2} + \frac{t_{\mathsf{mix}}}{\mu_{\min}(1-\gamma)}$$



- reflects cost taken to reach steady state
- one-time expense (almost independent of ε)
  — it becomes amortized as algorithm runs
- can be improved with the aid of variance reduction (Li et al. '20)
  — *prior art:* $\frac{t_{\mathsf{mix}}}{\mu_{\min}^2(1-\gamma)^5\varepsilon^2}$ (Qu & Wierman '20)

Model-free RL

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Recap: offline RL / batch RL

Historical dataset $\mathcal{D} = \{(s^{(i)}, a^{(i)}, s'^{(i)})\}$**:** N independent copies of

$$s \sim \rho^{\mathsf{b}}, \qquad a \sim \pi^{\mathsf{b}}(\cdot \,|\, s), \qquad s' \sim P(\cdot \,|\, s, a)$$

for some state distribution ρ^{b} and behavior policy π^{b}

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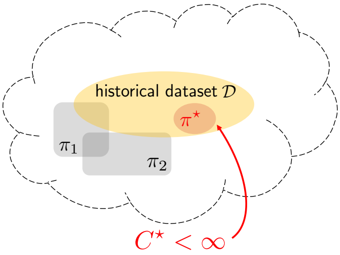
Single-policy concentrability

$$C^{\star} := \max_{s,a} \frac{d^{\pi^{\star}}(s,a)}{d^{\pi^{\mathsf{b}}}(s,a)} \geq 1$$

where d^{π}: occupancy distribution under π

- captures distributional shift
- allows for partial coverage



How to design offline model-free algorithms with optimal sample efficiency?

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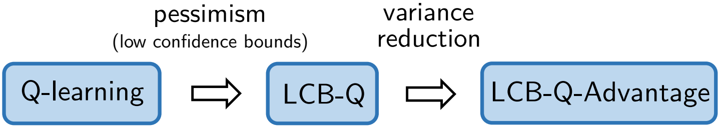

LCB-Q: Q-learning with LCB penalty

— *Shi et al. '22, Yan et al. '22*

$$Q_{t+1}(s_t, a_t) \leftarrow \underbrace{(1-\eta_t)Q_t(s_t, a_t) + \eta_t \mathcal{T}_t\left(Q_t\right)(s_t, a_t)}_{\text{classical Q-learning}} - \eta_t \underbrace{b_t(s_t, a_t)}_{\text{LCB penalty}}$$

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- $b_t(s, a)$: Hoeffding-style confidence bound
- pessimism in the face of uncertainty

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sample size: $\widetilde{O}\left(\frac{SC^\star}{(1-\gamma)^5\varepsilon^2}\right)$ $\Longrightarrow$ sub-optimal by a factor of $\frac{1}{(1-\gamma)^2}$

Issue: *large variability in stochastic update rules*

Q-learning with LCB and variance reduction

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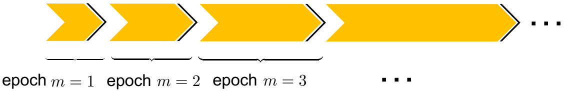

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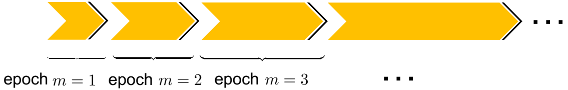
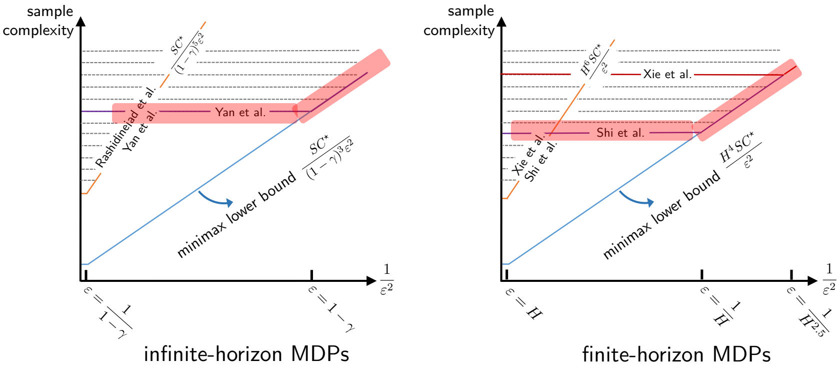


Theorem 5 (Yan, Li, Chen, Fan '22, Shi, Li, Wei, Chen, Chi '22)

For $\varepsilon \in (0, 1-\gamma]$, LCB-Q-Advantage *achieves* $V^\star(\rho) - V^{\widehat{\pi}}(\rho) \le \varepsilon$ *with optimal sample complexity* $\widetilde{O}\big(\frac{SC^\star}{(1-\gamma)^3\varepsilon^2}\big)$



Model-free offline RL attains sample optimality too!

— *with some burn-in cost though . . .*

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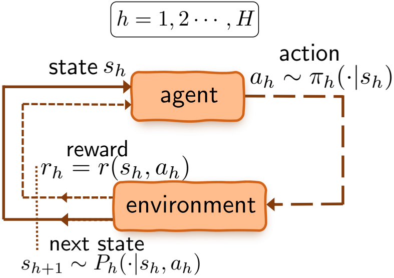
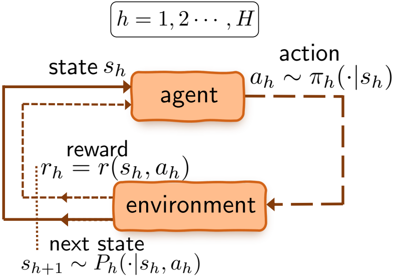
Finite-horizon MDPs



- H: horizon length
- $\mathcal{S}$: state space with size S
- $\mathcal{A}$: action space with size A
- $r_h(s_h, a_h) \in [0, 1]$: immediate reward in step h
- $\pi = \{\pi_h\}_{h=1}^{H}$: policy (or action selection rule)
- $P_h(\cdot \mid s, a)$: transition probabilities in step h

Finite-horizon MDPs



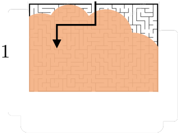
value function: $V_h^\pi(s) := \mathbb{E}\left[\sum_{t=h}^{H} r_h(s_h, a_h) \,\Big|\, s_h = s\right]$

Q-function: $Q_h^\pi(s, a) := \mathbb{E}\left[\sum_{t=h}^{H} r_h(s_h, a_h) \,\Big|\, s_h = s, a_h = a\right]$

Online RL: interacting with real environments

Sequentially execute MDP for K episodes, each consisting of H steps

episode 1



execute π^1 $\Longrightarrow$ $\{s_h^1, a_h^1, r_h^1\}_{h=1}^H$

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Sequentially execute MDP for K episodes, each consisting of H steps

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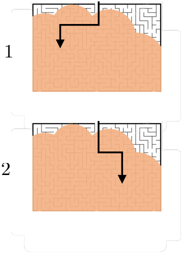


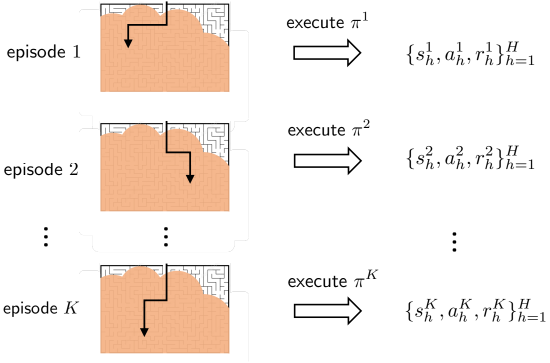
execute π^1 $\Longrightarrow$ $\{s_h^1, a_h^1, r_h^1\}_{h=1}^H$

episode 2

execute π^2 $\Longrightarrow$ $\{s_h^2, a_h^2, r_h^2\}_{h=1}^H$

Online RL: interacting with real environments

Sequentially execute MDP for K episodes, each consisting of H steps



Online RL: interacting with real environments

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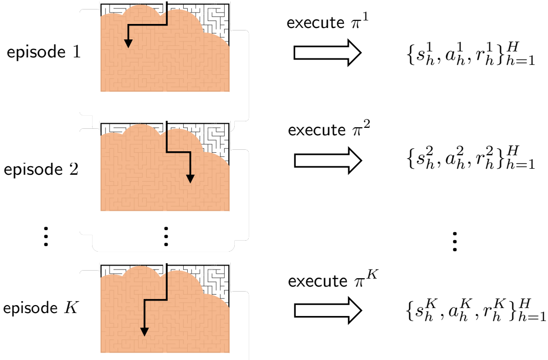
— *sample size:* $T = KH$



exploration (exploring unknowns) vs. **exploitation** (exploiting learned info)

Regret: gap between learned policy & optimal policy

adversary

learner

initial state s_1^1 ⇨ execute policy π^1

episode 1

Regret: gap between learned policy & optimal policy

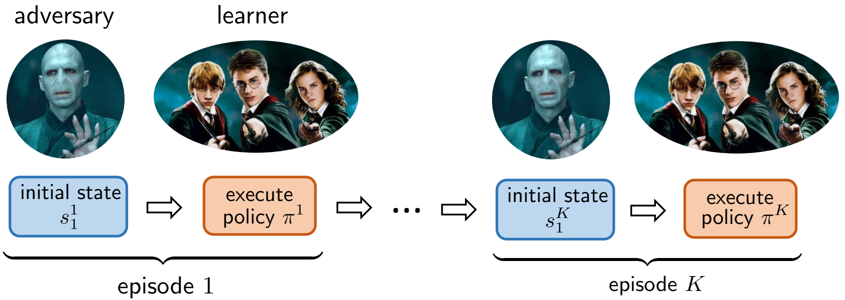
adversary

learner

initial state s_1^1 ⇒ execute policy π^1 ⇒ ... ⇒ initial state s_1^K ⇒ execute policy π^K

episode 1 ... episode K

Regret: gap between learned policy & optimal policy



Performance metric: given $\underbrace{\text{initial states } \{s_1^k\}_{k=1}^K}_{\text{chosen by nature/adversary}}$, define

$$\text{Regret}(T) := \sum_{k=1}^{K} \left(V_1^\star(s_1^k) - V_1^{\pi^k}(s_1^k) \right)$$

Lower bound

(Domingues et al. '21)

$$\text{Regret}(T) \gtrsim \sqrt{H^2SAT}$$

Existing algorithms

- UCB-VI: Azar et al. '17
- UBEV: Dann et al. '17
- UCB-Q-Hoeffding: Jin et al. '18
- **UCB-Q-Bernstein: Jin et al. '18**
- UCB2-Q-Bernstein: Bai et al. '19
- EULER: Zanette et al. '19
- **UCB-Q-Advantage: Zhang et al. '20**
- UCB-M-Q: Menard et al. '21
- **Q-EarlySettled-Advantage: Li et al. '21**

Which model-free algorithms are sample-efficient for online RL?

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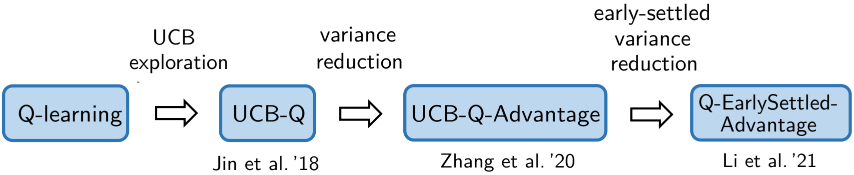

Q-learning with UCB exploration (Jin et al., 2018)

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- employ variance reduction to help acclerate convergence

UCB-Q-Advantage is asymptotically regret-optimal

Issue: *high burn-in cost* $O(S^6A^4H^{28})$

UCB-Q with variance reduction and early settlement

One additional key idea: early settlement of the reference as soon as it reaches a reasonable quality

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Theorem 6 (Li, Shi, Chen, Gu, Chi '21)

With high prob., Q-EarlySettled-Advantage *achieves*

$$\mathsf{Regret}(T) \leq \widetilde{O}\left(\sqrt{H^2SAT} + H^6SA\right)$$

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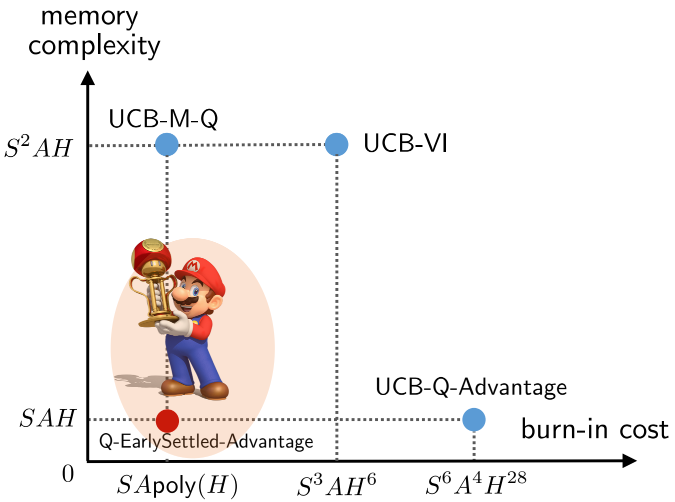
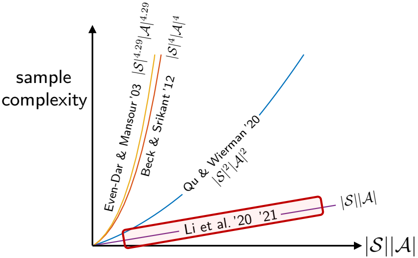
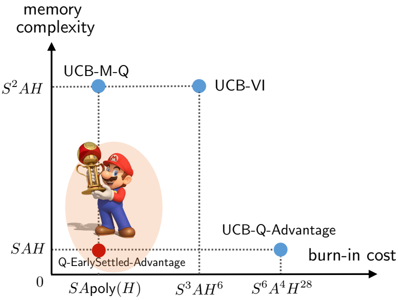
- regret-optimal with $\underbrace{\text{near-minimal burn-in cost}}_{SA\mathrm{poly}(H)}$ in S and A
- memory-efficient $O(SAH)$
- computationally efficient: runtime $O(T)$

Comparisons of regret-optimal algorithms



Summary of this part





Model-free RL can achieve memory efficiency, computational efficiency, and sample efficiency at once!

— *with some burn-in cost though*

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