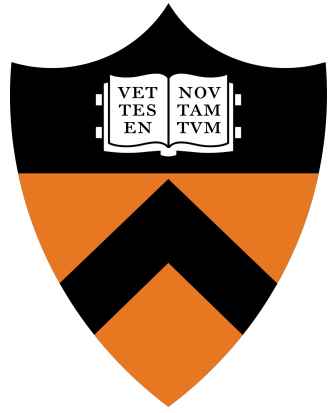
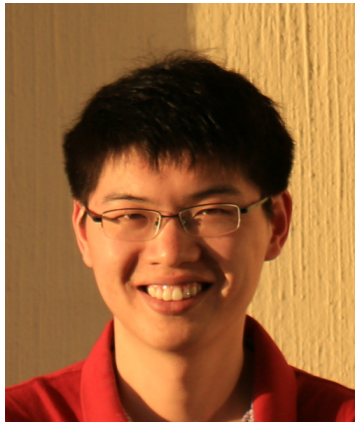
Inference and Uncertainty Quantification for Noisy Matrix Completion

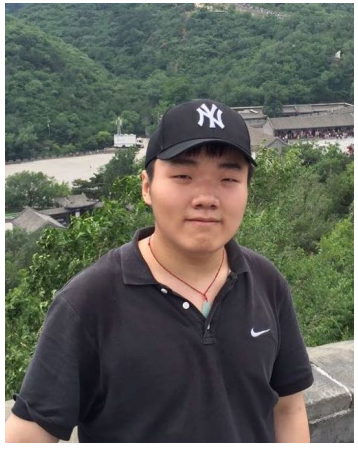


Yuxin Chen

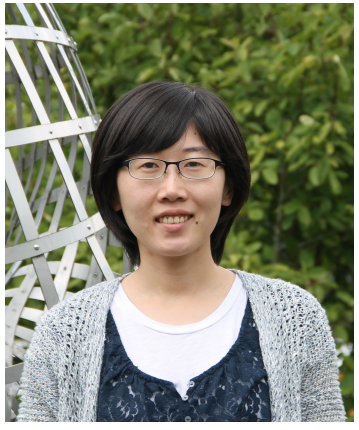
EE, Princeton University



Cong Ma
Princeton ORFE



Yuling Yan
Princeton ORFE



Yuejie Chi
CMU ECE

Jianqing Fan
Princeton ORFE

Low-rank matrix completion

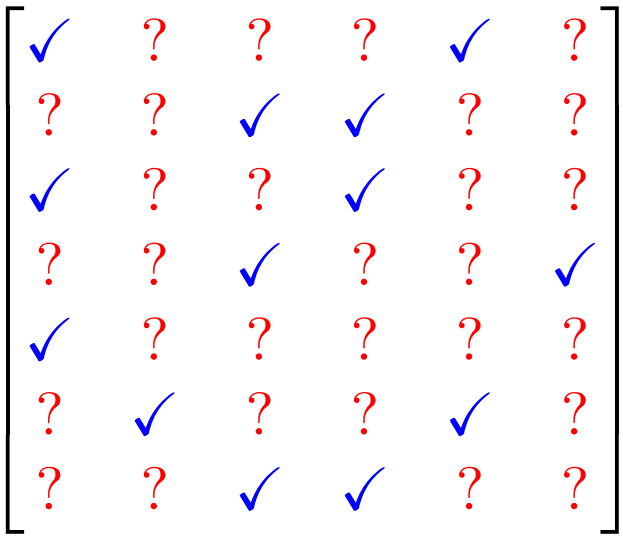
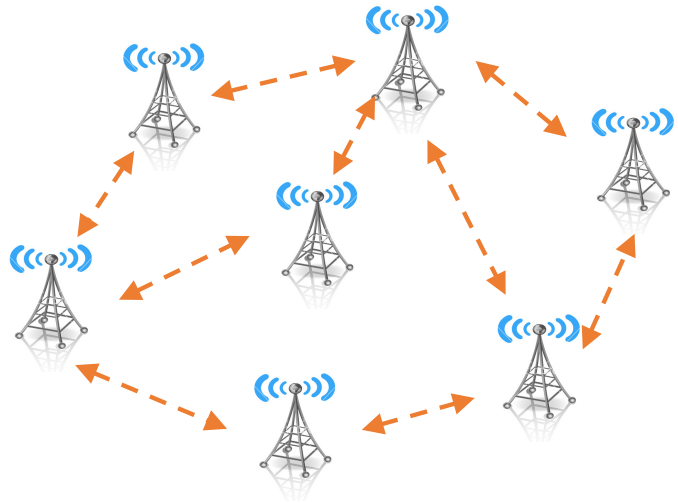


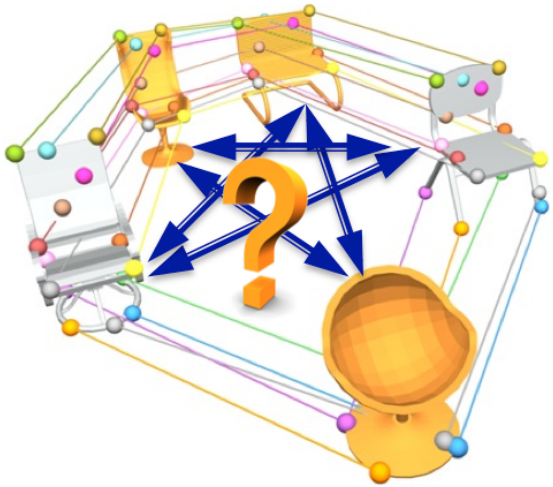
figure credit: E. J. Candès

Given partial samples of a low-rank matrix $\boldsymbol{M}^\star$, fill in missing entries

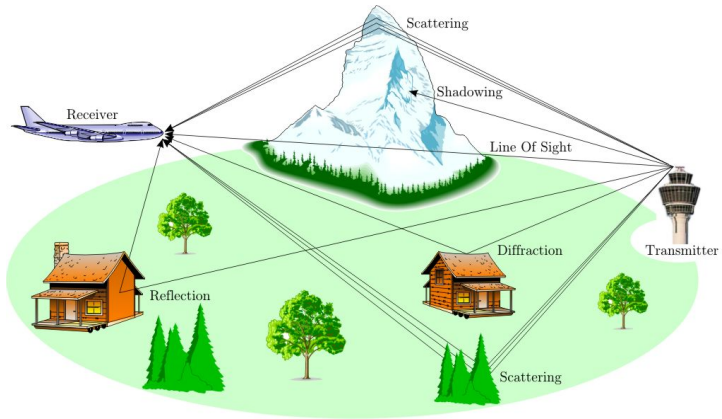
recommendation systems



localization



shape matching



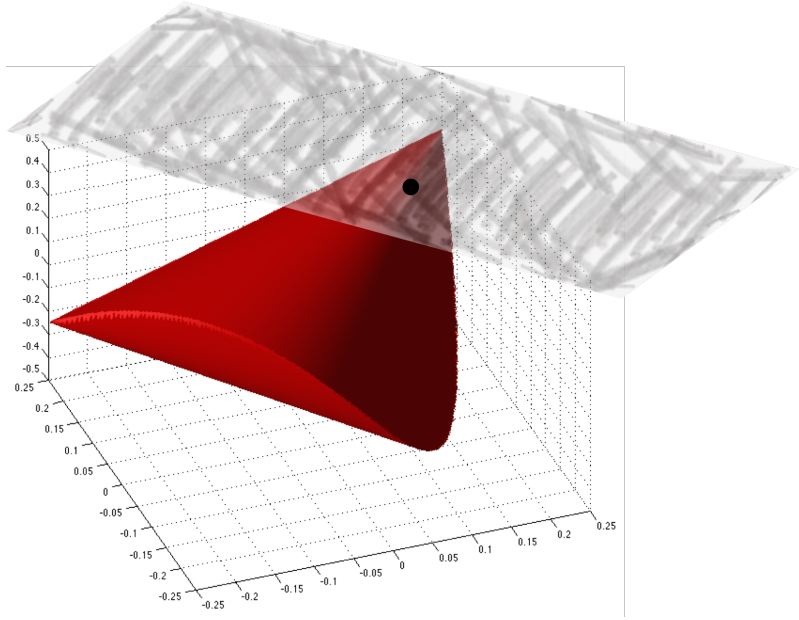
channel estimation

Estimation via convex relaxation

$$\underset{Z}{\text{minimize}} \quad \underbrace{f(Z; \text{data})}_{\text{empirical loss}} + \lambda \|Z\|_*$$

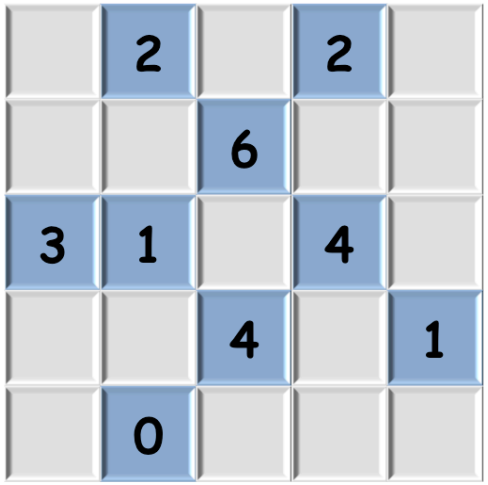
low-rank matrix

Composition C by Piet Mondrian

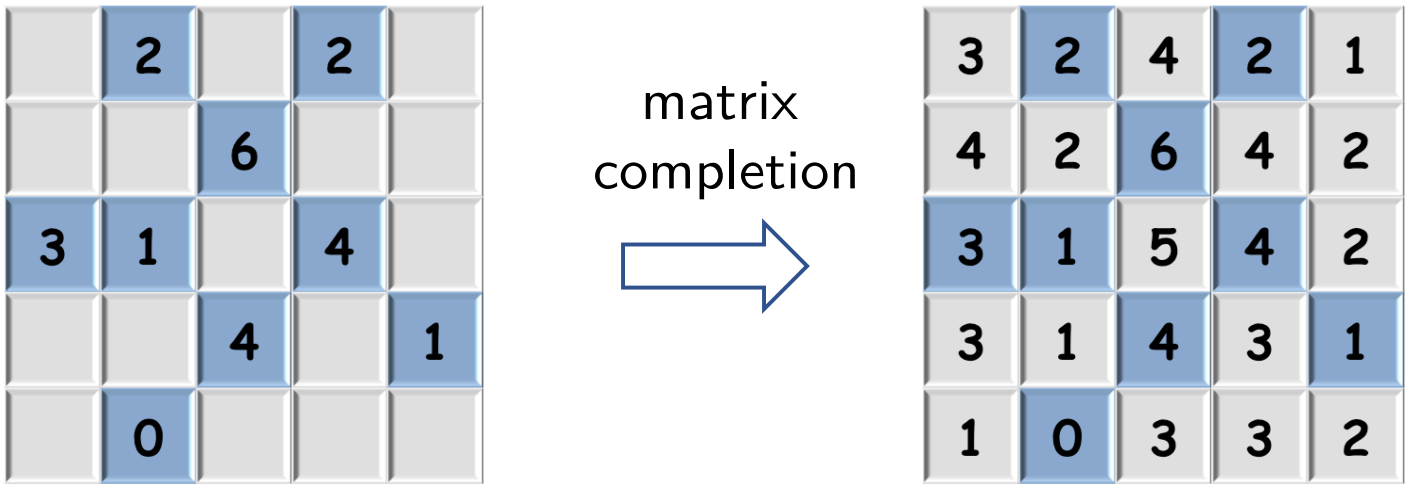


semidefinite relaxation

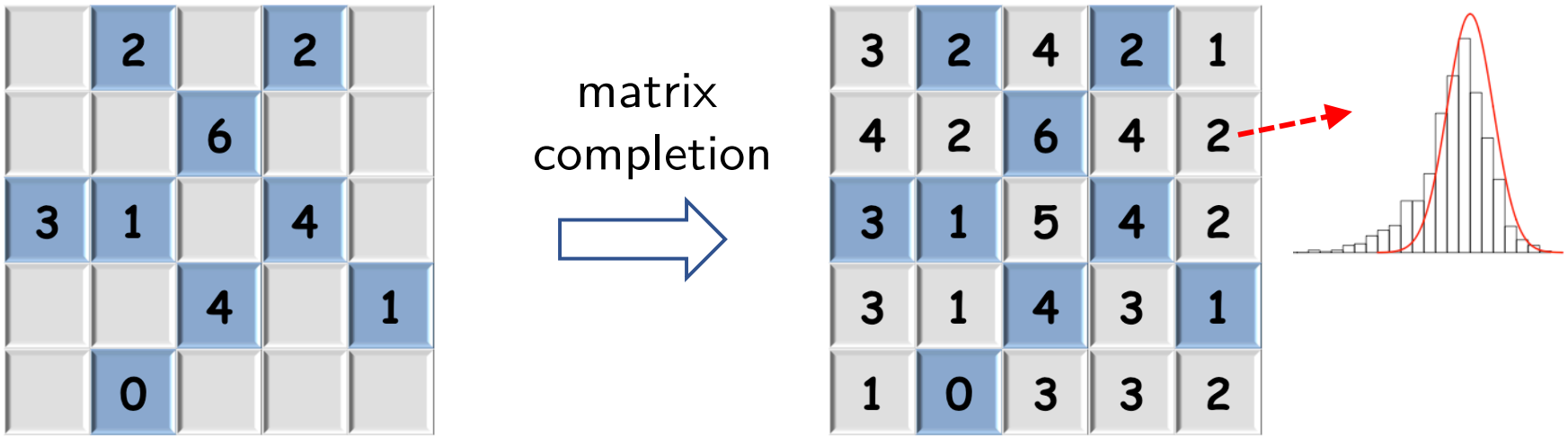
One step further: reasoning about uncertainty?



One step further: reasoning about uncertainty?



One step further: reasoning about uncertainty?

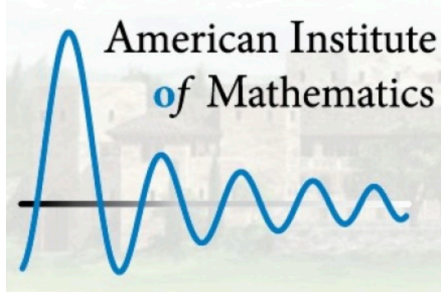


How to assess uncertainty, or "confidence", of obtained estimates due to imperfect data acquisition?

- noise
- incomplete measurements
- ...

INFERENCE IN HIGH DIMENSIONAL REGRESSION

organized by
Peter Buehlmann, Andrea Montanari, and Jonathan Taylor



(3) *Confidence intervals for matrix completion.* In matrix completion, the data analyst is given a large data matrix with a number of missing entries. In many interesting applications (e.g. to collaborative filtering) it is indeed the case that the vast majority of entries is missing. In order to fill the missing entries, the assumption is made that the underlying –unknown– matrix has a low-rank structure.

Substantial work has been devoted to methods for computing point estimates of the missing entries. In applications, it would be very interesting to compute confidence intervals as well. This requires developing distributional characterizations of standard matrix completion methods.

Challenges

$$\boldsymbol{M}^{\mathsf{cvx}} \triangleq \arg\min_{\boldsymbol{Z}} \underbrace{f(\boldsymbol{Z}; \mathsf{data})}_{\text{empirical loss}} + \lambda \|\boldsymbol{Z}\|_*$$

- convex estimate $\boldsymbol{M}^{\mathsf{cvx}}$ is biased towards small norm

Challenges

$$\boldsymbol{M}^{\mathsf{cvx}} \triangleq \arg\min_{\boldsymbol{Z}} \underbrace{f(\boldsymbol{Z}; \mathsf{data})}_{\text{empirical loss}} + \lambda \|\boldsymbol{Z}\|_*$$

- convex estimate $\boldsymbol{M}^{\mathsf{cvx}}$ is biased towards small norm
- very challenging to pin down distributions of obtained estimates

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- convex estimate $\boldsymbol{M}^{\mathsf{cvx}}$ is biased towards small norm
- very challenging to pin down distributions of obtained estimates
- existing estimation error bounds are highly sub-optimal
  $\longrightarrow$ overly wide confidence intervals

This talk

1. **Stability:** order-wise optimal ℓ_2 estimation error bounds

This talk

1. **Stability:** order-wise optimal ℓ_2 estimation error bounds

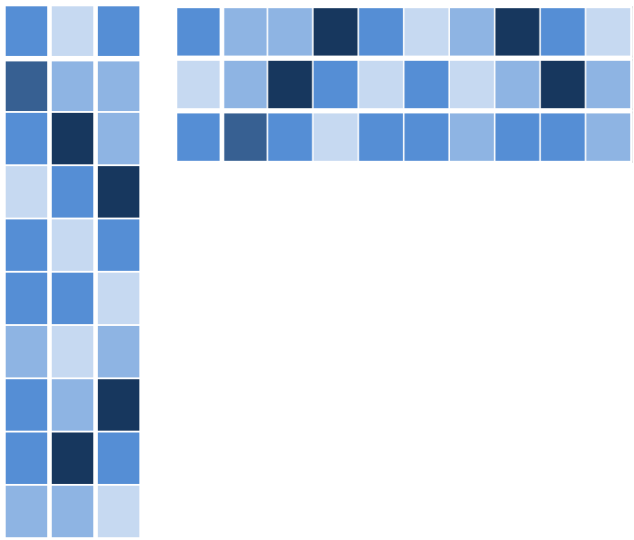
2. **Uncertainty quantification:** optimal entrywise confidence intervals (including both rates & pre-constants)

Stability analysis

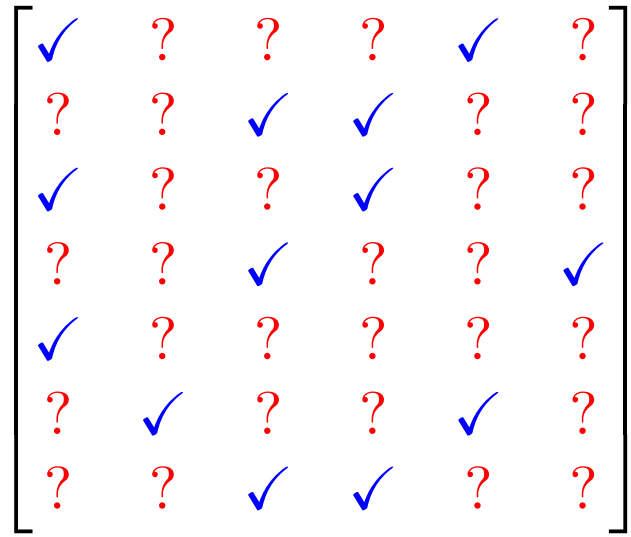
Noisy low-rank matrix completion

observations: $M_{i,j} = M^{\star}_{i,j} + \text{noise}, \quad (i,j) \in \Omega$

goal: estimate $\boldsymbol{M}^{\star}$



unknown rank-r matrix $\boldsymbol{M}^{\star} \in \mathbb{R}^{n \times n}$



sampling set Ω

Noisy low-rank matrix completion

observations: $M_{i,j} = M^{\star}_{i,j} + \text{noise}, \quad (i,j) \in \Omega$

goal: estimate $\boldsymbol{M}^{\star}$

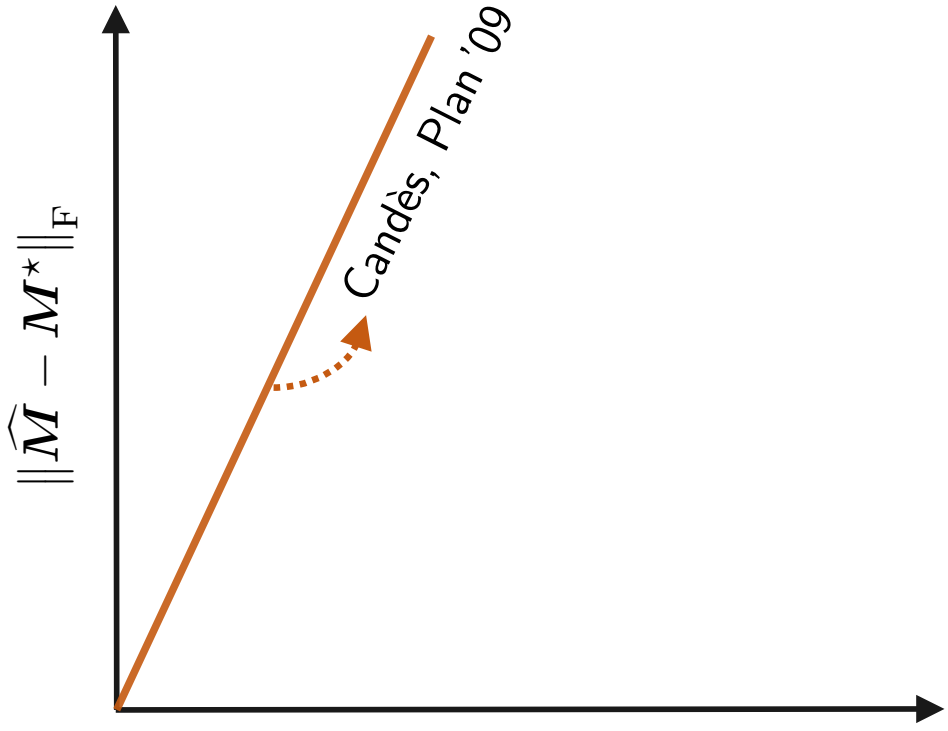
convex relaxation:

$$\underset{\boldsymbol{Z} \in \mathbb{R}^{n \times n}}{\text{minimize}} \quad \underbrace{\sum_{(i,j) \in \Omega} \left(Z_{i,j} - M_{i,j}\right)^2}_{\text{squared loss}} + \lambda \|\boldsymbol{Z}\|_*$$

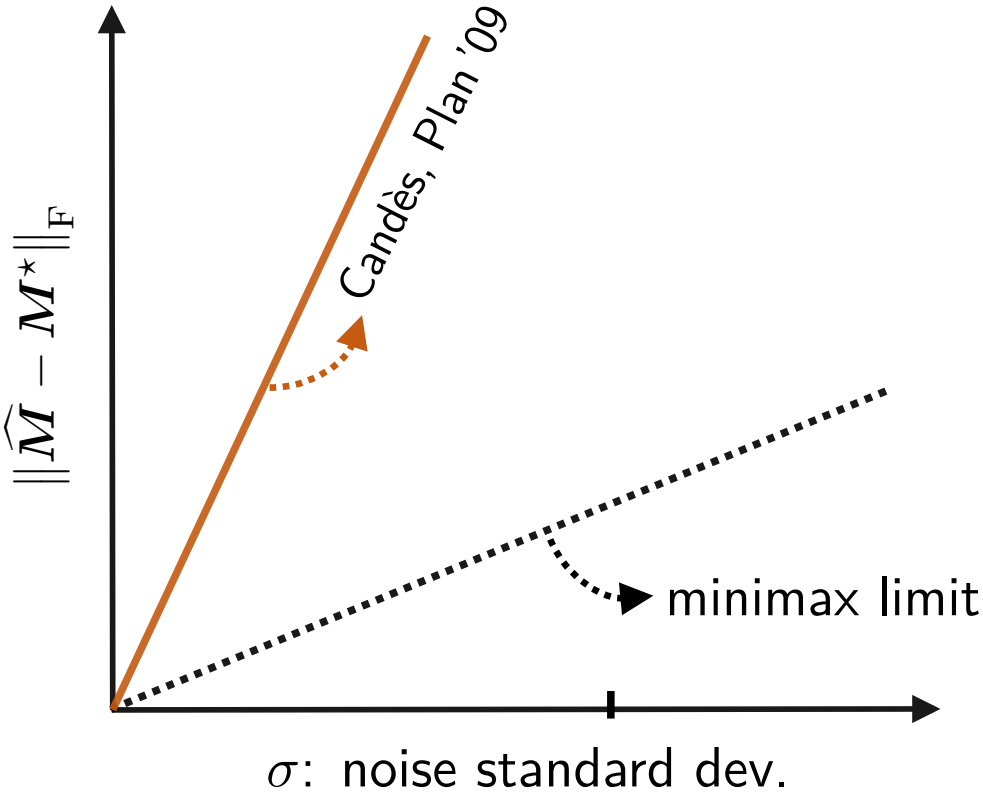
Prior statistical guarantees for convex relaxation

- **random sampling:** each $(i, j) \in \Omega$ with prob. p
- **random noise:** i.i.d. sub-Gaussian noise with variance σ^2
- true matrix $\boldsymbol{M}^\star \in \mathbb{R}^{n \times n}$: rank $r = O(1)$, incoherent, ...

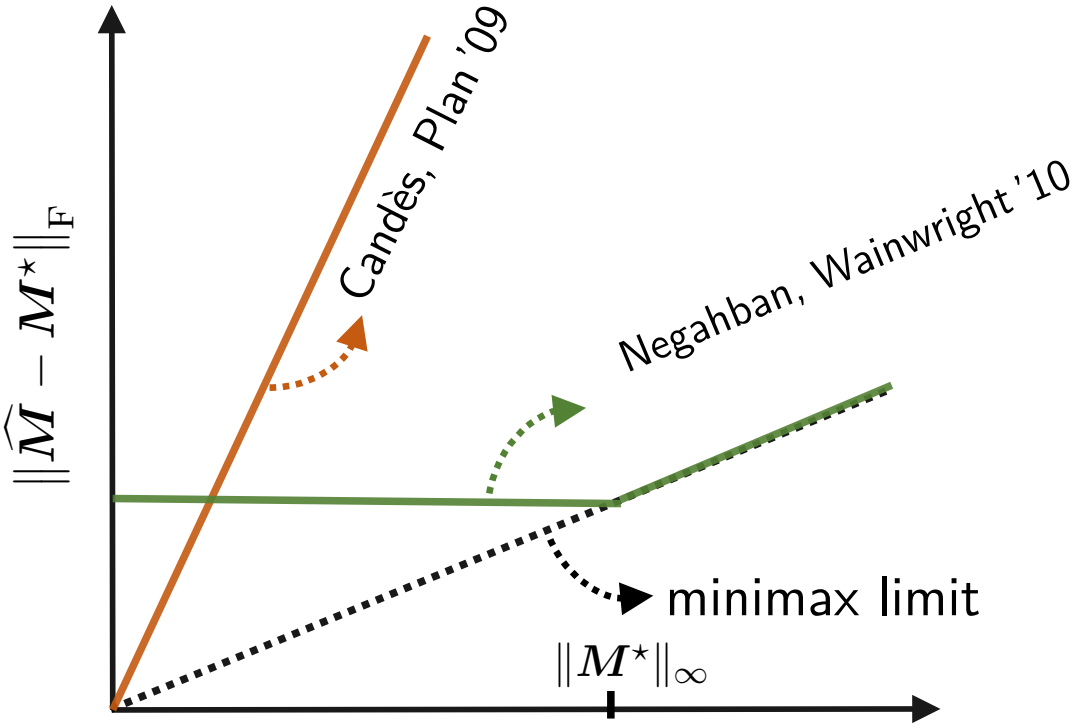
Candès, Plan '09	$\sigma n^{1.5}$



minimax limit	$\sigma\sqrt{n/p}$
Candès, Plan '09	$\sigma n^{1.5}$



minimax limit	$\sigma\sqrt{n/p}$
Candès, Plan '09	$\sigma n^{1.5}$
Negahban, Wainwright '10	$\max\{\sigma, \|\boldsymbol{M}^\star\|_\infty\}\sqrt{n/p}$



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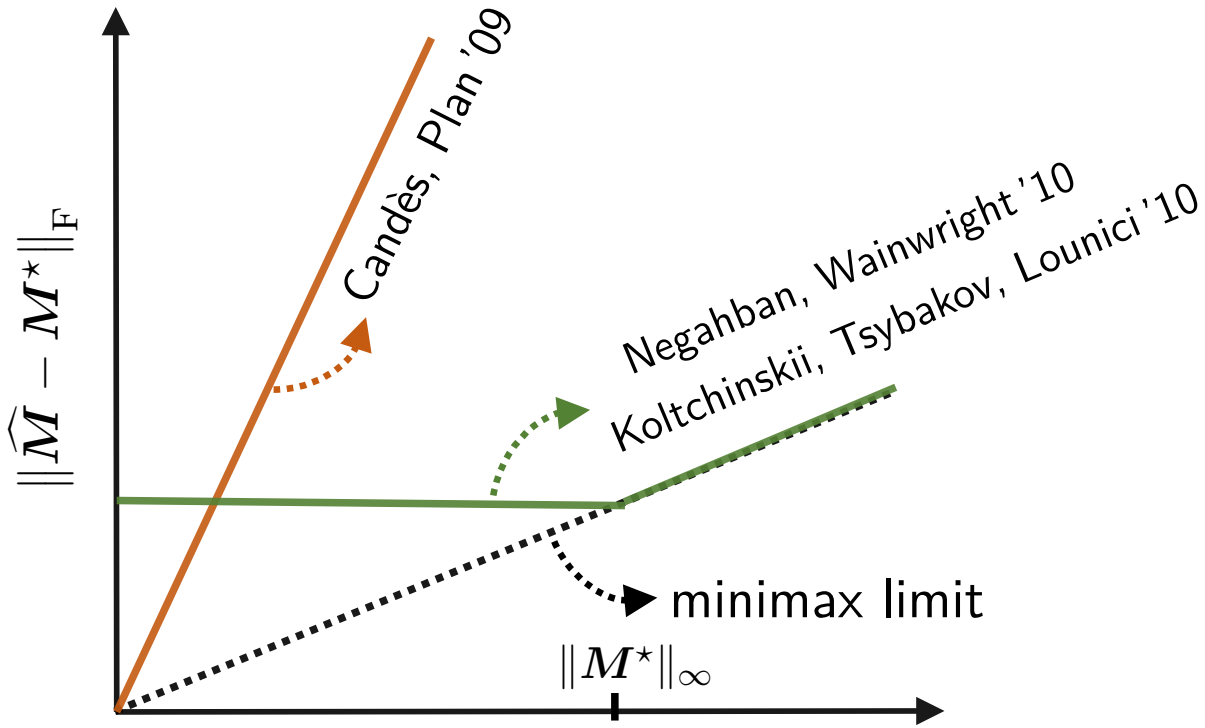
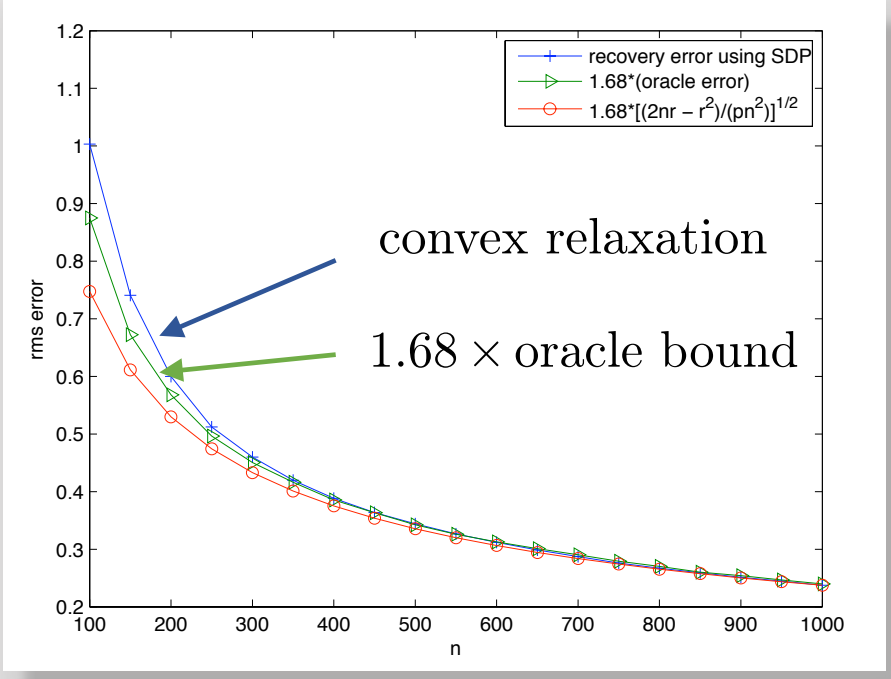
Koltchinskii, Tsybakov, Lounici '10	$\max\{\sigma, \|\boldsymbol{M}^\star\|_\infty\}\sqrt{n/p}$



Matrix Completion with Noise

Emmanuel J. Candès and Yaniv Plan



Existing theory for convex relaxation does not match practice . . .

Matrix Completion with Noise

Emmanuel J. Candès and Yaniv Plan

with adversarial noise. Consequently, our analysis looses a $\sqrt{n}$ factor vis a vis an optimal bound that is achievable via the help of an oracle.

Existing theory for convex relaxation does not match practice . . .

What are the roadblocks?

Strategy: $\boldsymbol{M}^{\text{cvx}}$ is optimizer if $\underbrace{\text{there exists } \boldsymbol{W}}_{\text{dual certificate}}$ s.t.

$(\boldsymbol{M}^{\text{cvx}}, \boldsymbol{W})$ obeys KKT optimality condition

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David Gross

- **noiseless case**: $\underbrace{\boldsymbol{M}^{\text{cvx}} \leftarrow \boldsymbol{M}^{\star}}_{\text{exact recovery}}$; $\boldsymbol{W} \leftarrow$ golfing scheme

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David Gross

- **noiseless case**: $\underbrace{\boldsymbol{M}^{\text{cvx}} \leftarrow \boldsymbol{M}^{\star}}_{\text{exact recovery}}$; $\boldsymbol{W} \leftarrow$ golfing scheme
- **noisy case**: $\boldsymbol{M}^{\text{cvx}}$ is very complicated, hard to construct $\boldsymbol{W}$ ...

dual certification (golfing scheme)

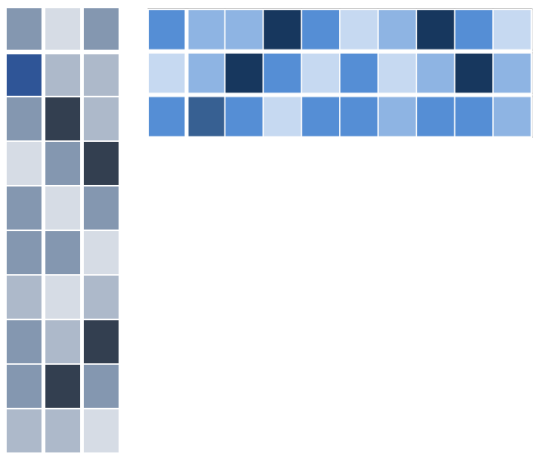
dual certification (golfing scheme)

nonconvex optimization

A detour: nonconvex optimization

Burer–Monteiro: represent $\boldsymbol{Z}$ by $\boldsymbol{X}\boldsymbol{Y}^\top$ with $\underbrace{\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n \times r}}_{\text{low-rank factors}}$

$\boldsymbol{X}$ $\quad$ $\boldsymbol{Y}^\top$



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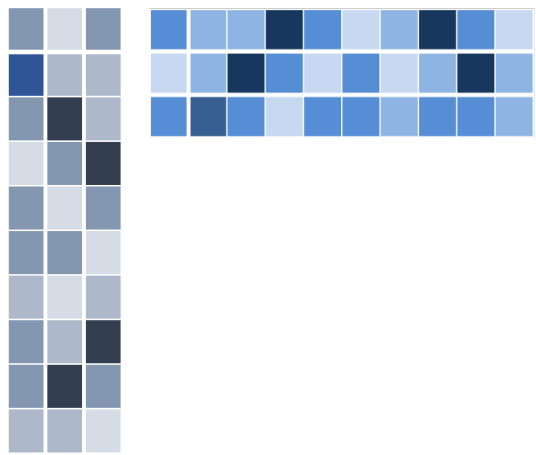

$$\underset{\boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^{n\times r}}{\text{minimize}} \quad f(\boldsymbol{X},\boldsymbol{Y}) = \underbrace{\sum_{(i,j)\in\Omega} \left[\left(\boldsymbol{X}\boldsymbol{Y}^\top\right)_{i,j} - M_{i,j}\right]^2}_{\text{squared loss}} + \text{reg}(\boldsymbol{X},\boldsymbol{Y})$$

A detour: nonconvex optimization

- Burer, Monteiro '03
- Rennie, Srebro '05
- Keshavan, Montanari, Oh '09 '10
- Jain, Netrapalli, Sanghavi '12
- Hardt '13
- Sun, Luo '14
- Chen, Wainwright '15
- Tu, Boczar, Simchowitz, Soltanolkotabi, Recht '15
- Zhao, Wang, Liu '15
- Zheng, Lafferty '16
- Yi, Park, Chen, Caramanis '16
- Ge, Lee, Ma '16
- Ge, Jin, Zheng '17
- Ma, Wang, Chi, Chen '17
- Chen, Li '18
- Chen, Liu, Li '19
- ...

A detour: nonconvex optimization

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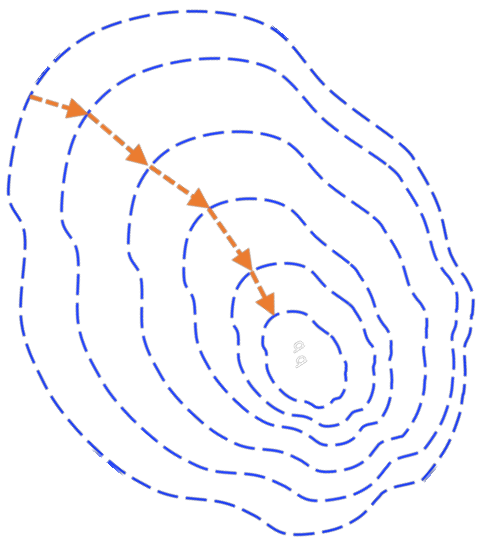


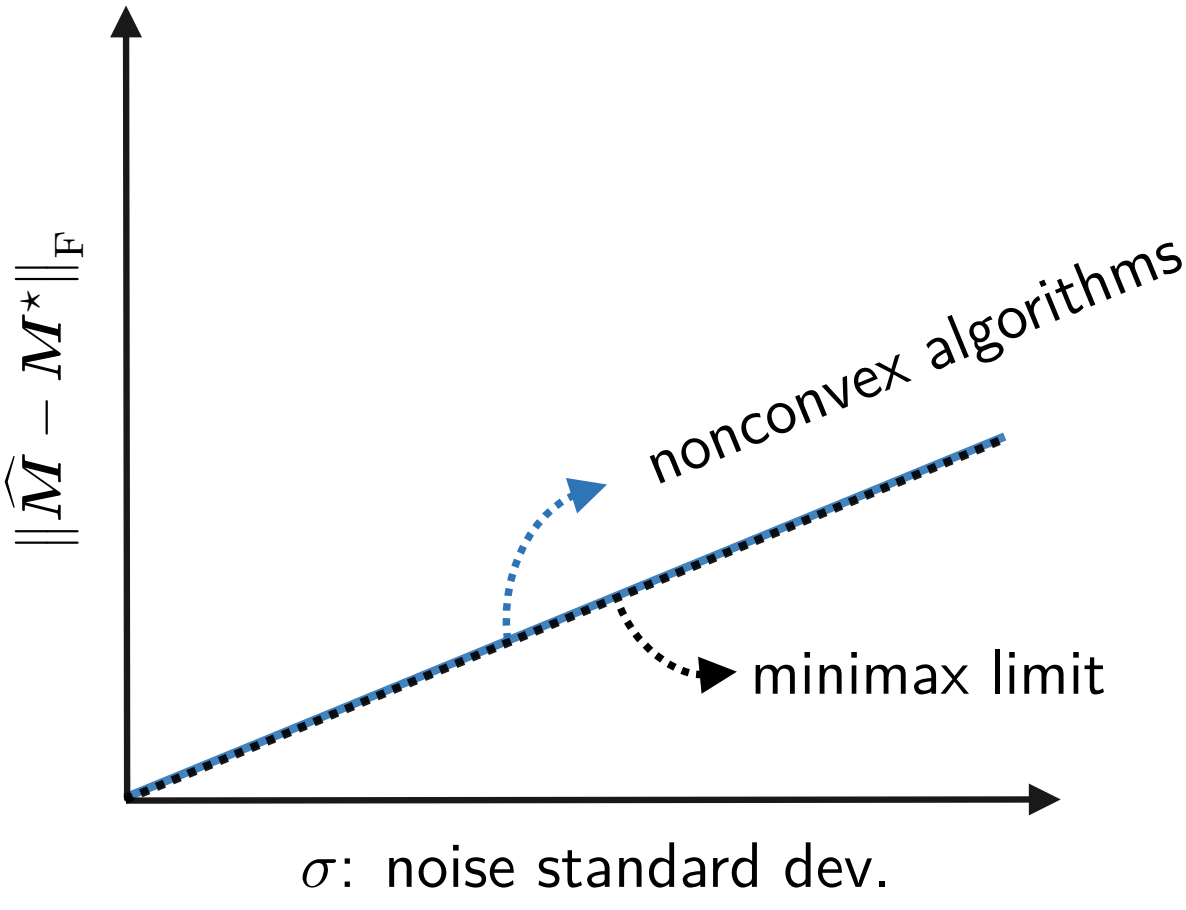
- **suitable initialization:** $(\boldsymbol{X}^0, \boldsymbol{Y}^0)$
- **gradient descent:** for $t = 0, 1, \ldots$

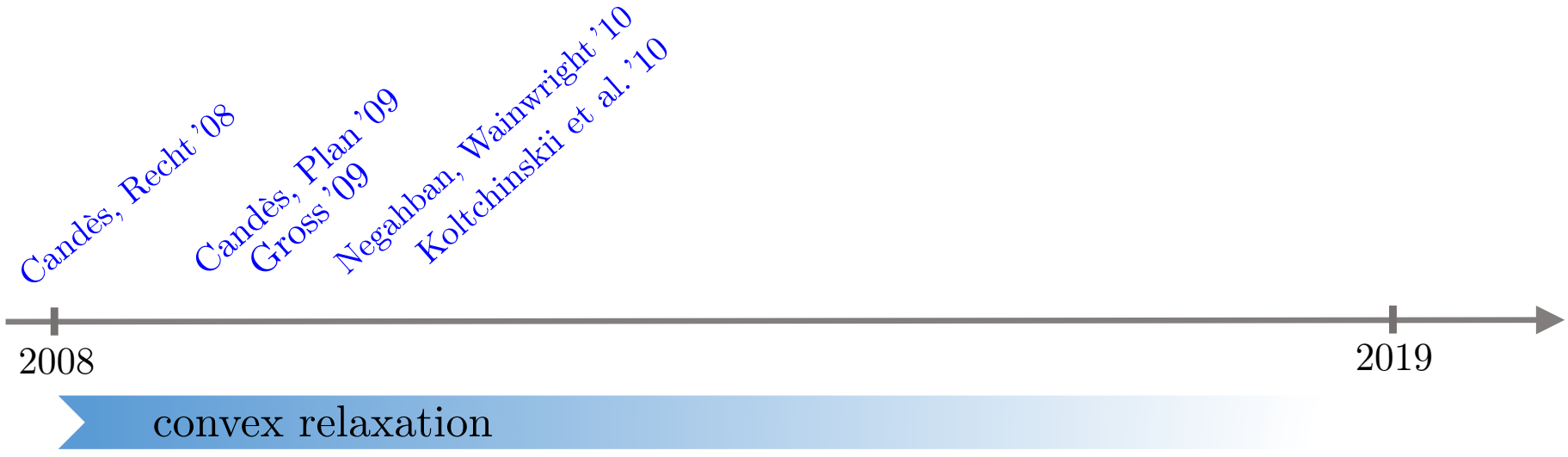
$$\boldsymbol{X}^{t+1} = \boldsymbol{X}^t - \eta_t \nabla_{\boldsymbol{X}} f(\boldsymbol{X}^t, \boldsymbol{Y}^t)$$
$$\boldsymbol{Y}^{t+1} = \boldsymbol{Y}^t - \eta_t \nabla_{\boldsymbol{Y}} f(\boldsymbol{X}^t, \boldsymbol{Y}^t)$$

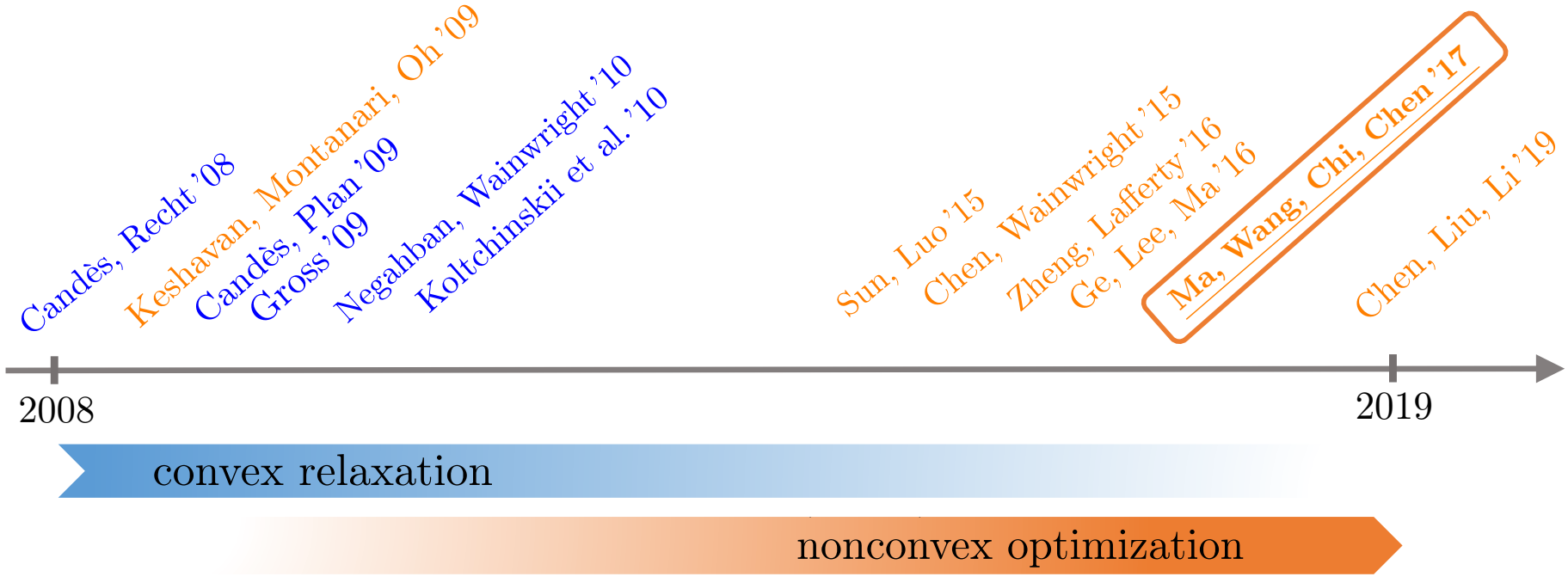
A detour: nonconvex optimization

- **random sampling:** each $(i, j) \in \Omega$ with prob. p
- **random noise:** i.i.d. sub-Gaussian with variance σ^2 (not too large)
- true matrix $\boldsymbol{M}^\star \in \mathbb{R}^{n \times n}$: $r = O(1)$, incoherent, ...

minimax limit	$\sigma\sqrt{n/p}$
nonconvex algorithms	$\sigma\sqrt{n/p}$ (optimal!)




Candès, Recht '08
Candès, Plan '09
Gross '09
Negahban, Wainwright '10
Koltchinskii et al. '10
2008
2019
convex relaxation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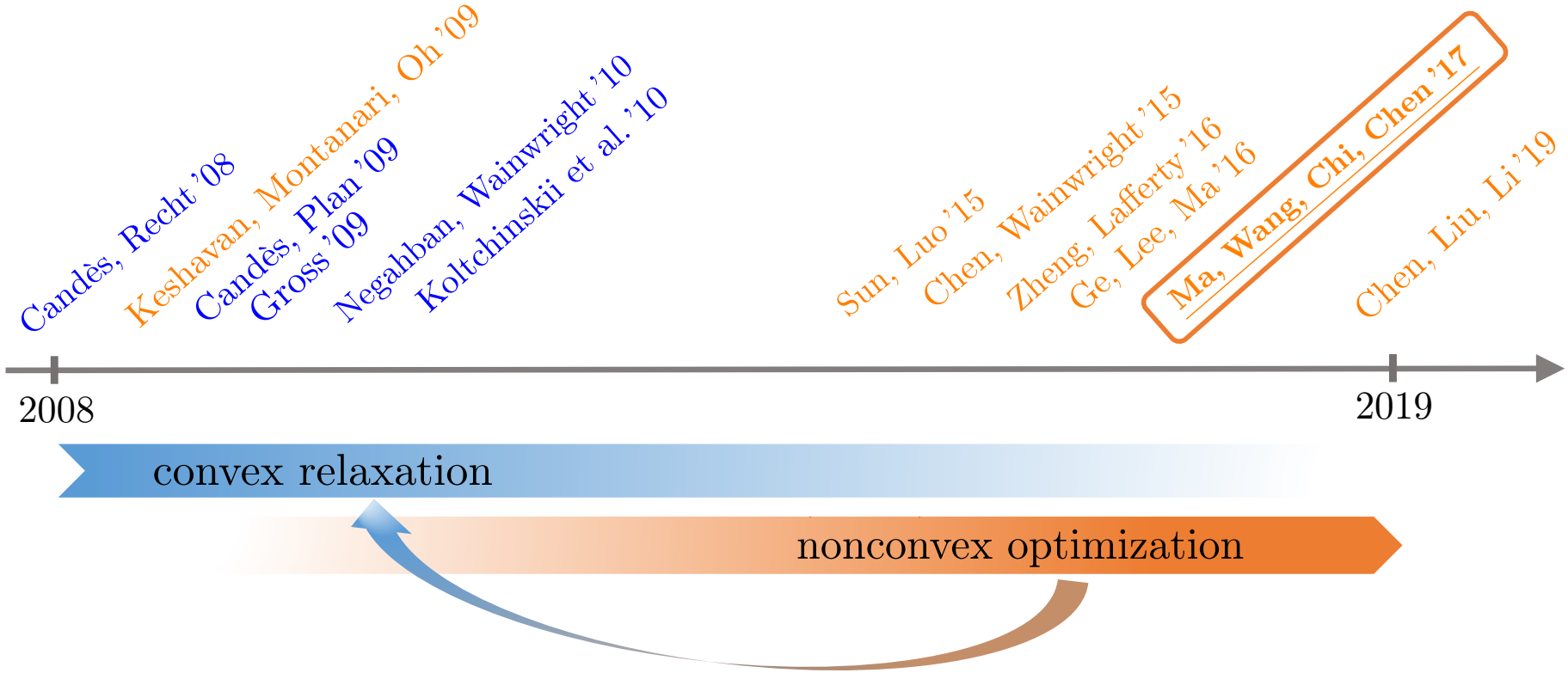
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convex relaxation
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A motivating experiment

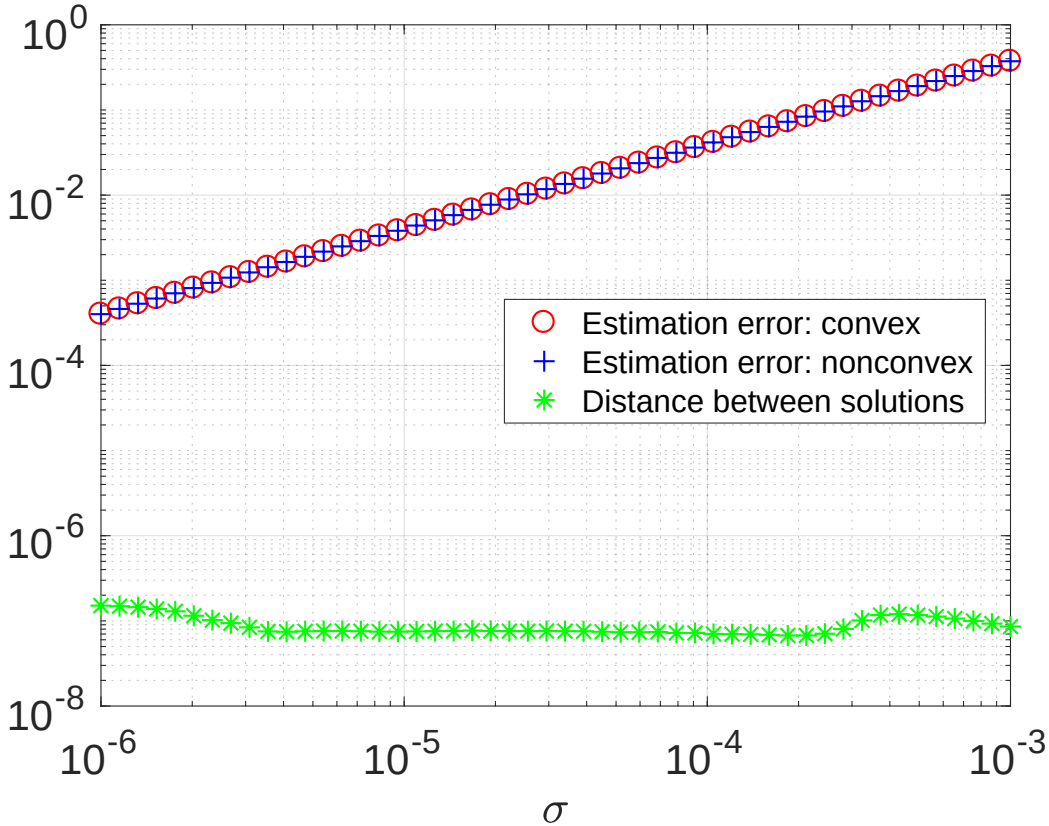
convex: $\underset{\boldsymbol{Z}\in\mathbb{R}^{n\times n}}{\text{minimize}} \sum_{(i,j)\in\Omega} \left(Z_{i,j} - M_{i,j}\right)^2 + \lambda\|\boldsymbol{Z}\|_*$

nonconvex: $\underset{\boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^{n\times r}}{\text{minimize}} \sum_{(i,j)\in\Omega} \left[\left(\boldsymbol{X}\boldsymbol{Y}^\top\right)_{i,j} - M_{i,j}\right]^2 + \underbrace{\frac{\lambda}{2}\|\boldsymbol{X}\|_{\mathrm{F}}^2 + \frac{\lambda}{2}\|\boldsymbol{Y}\|_{\mathrm{F}}^2}_{\mathsf{reg}(\boldsymbol{X},\boldsymbol{Y})}$

— $\|\boldsymbol{Z}\|_* = \min_{\boldsymbol{Z}=\boldsymbol{X}\boldsymbol{Y}^\top} \frac{1}{2}\|\boldsymbol{X}\|_{\mathrm{F}}^2 + \frac{1}{2}\|\boldsymbol{Y}\|_{\mathrm{F}}^2$

A motivating experiment

$$n = 1000,\ r = 5,\ p = 0.2,\ \lambda = 5\sigma\sqrt{np}$$



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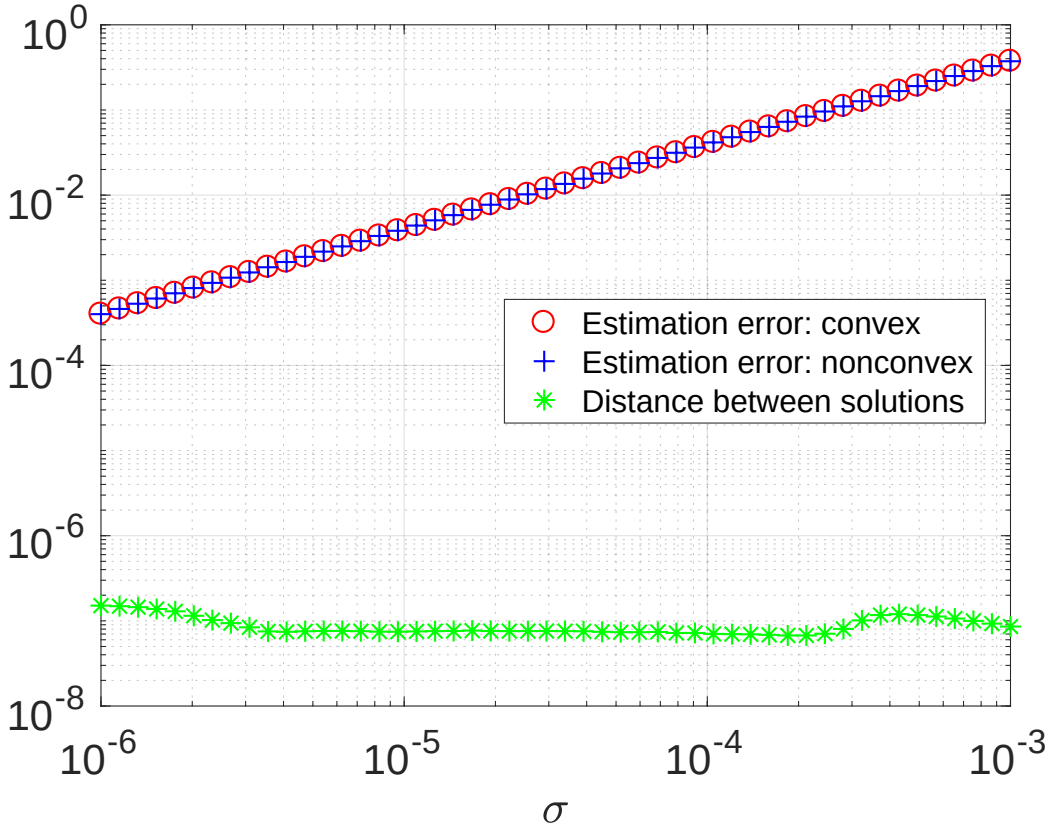

Convex and nonconvex solutions are exceedingly close!

convex

nonconvex

$$\Downarrow$$

$$\text{stability}\Big(\text{convex}\Big) \approx \text{stability}\Big(\text{nonconvex}\Big)$$

Main results: $r = O(1)$

- **random sampling:** each $(i, j) \in \Omega$ with prob. $p \gtrsim \frac{\log^3 n}{n}$
- **random noise:** i.i.d. sub-Gaussian with variance σ^2 (not too large)
- true matrix $\boldsymbol{M}^\star \in \mathbb{R}^{n \times n}$: $r = O(1)$, incoherent, well-conditioned

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$$\underset{\boldsymbol{Z} \in \mathbb{R}^{n \times n}}{\text{minimize}} \quad \sum_{(i,j) \in \Omega} \left(Z_{i,j} - M_{i,j}\right)^2 + \lambda \|\boldsymbol{Z}\|_* \qquad (\lambda \asymp \sigma \sqrt{np})$$

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Theorem 1 (Chen, Chi, Fan, Ma, Yan '19)

With high prob., any minimizer $\boldsymbol{M}^{\text{cvx}}$ of convex program obeys

1. *$\boldsymbol{M}^{\text{cvx}}$ is nearly rank-r*

$$\|\boldsymbol{M}^{\text{cvx}} - \text{proj}_{\textit{rank-}r}(\boldsymbol{M}^{\text{cvx}})\|_{\text{F}} \ll \frac{1}{n^5}\cdot\sigma\sqrt{\frac{n}{p}}$$

Main results: $r = O(1)$

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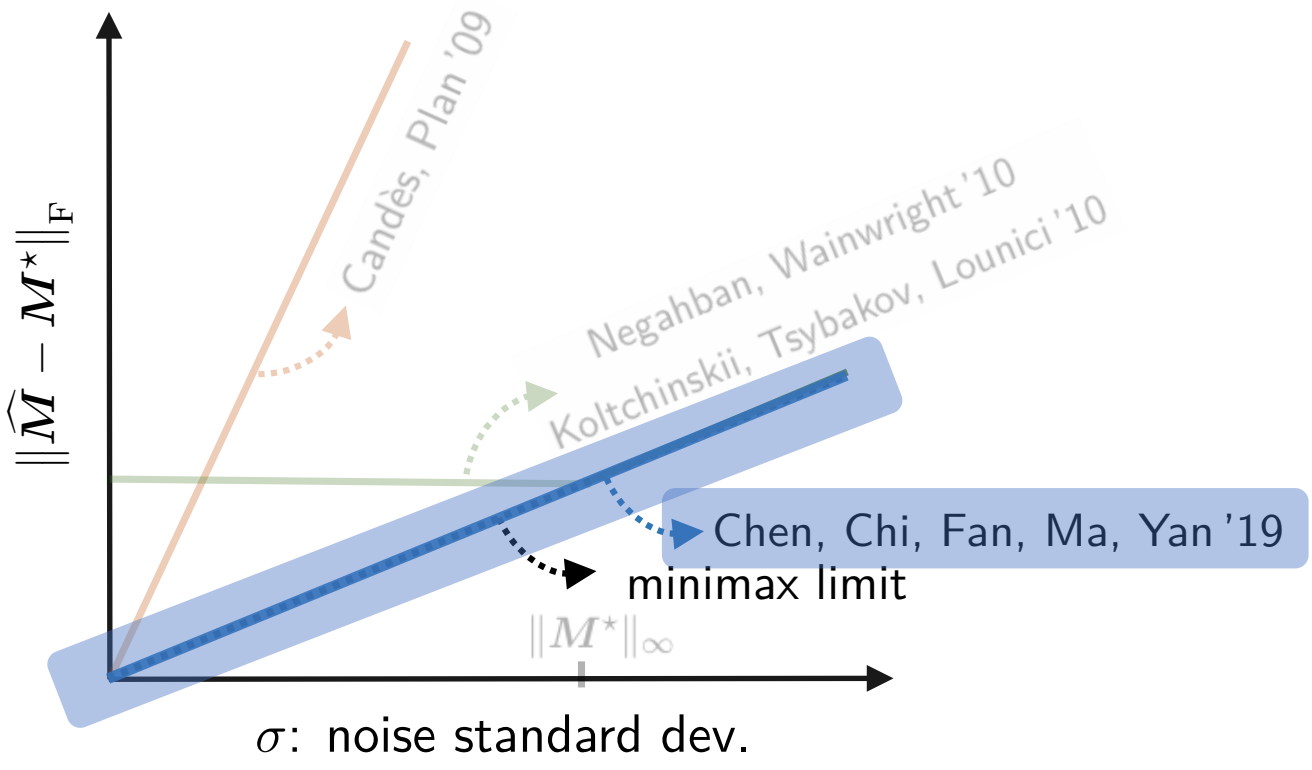
With high prob., any minimizer $\boldsymbol{M}^{\text{cvx}}$ of convex program obeys

1. *$\boldsymbol{M}^{\text{cvx}}$ is nearly rank-r*
2. $$\left\|\boldsymbol{M}^{\text{cvx}} - \boldsymbol{M}^\star\right\|_{\text{F}} \lesssim \sigma\sqrt{\frac{n}{p}}$$

$\left\|\boldsymbol{M}^{\mathsf{cvx}} - \boldsymbol{M}^{\star}\right\|_{\mathrm{F}} \lesssim \sigma\sqrt{\frac{n}{p}}$: minimax optimal when $r = O(1)$



Implicit regularization

No need to enforce spikiness constraint as in Negahban & Wainwright

$$\underset{\|\boldsymbol{Z}\|_\infty \leq \alpha}{\text{minimize}} \quad \sum_{(i,j)\in\Omega} \left(Z_{i,j} - M_{i,j}\right)^2 + \lambda \|\boldsymbol{Z}\|_* \qquad \text{(Negahban et al.)}$$

- convex programming automatically controls spikiness of solutions

Main results: general case

- **random sampling:** each $(i,j) \in \Omega$ with prob. $p \gtrsim \frac{r^2 \log^3 n}{n}$
- **random noise:** i.i.d. sub-Gaussian with variance σ^2 (not too large)
- true matrix $\boldsymbol{M}^\star \in \mathbb{R}^{n \times n}$: incoherent, well-conditioned

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Theorem 2 (Chen, Chi, Fan, Ma, Yan '19)

With high prob., any minimizer $\boldsymbol{M}^{\mathsf{cvx}}$ of convex program obeys

1. *$\boldsymbol{M}^{\mathsf{cvx}}$ is nearly rank-r*
2. $$\left\|\boldsymbol{M}^{\mathsf{cvx}} - \boldsymbol{M}^\star\right\|_{\mathrm{F}} \lesssim \frac{\sigma}{\sigma_{\min}(\boldsymbol{M}^\star)} \sqrt{\frac{n}{p}}\, \left\|\boldsymbol{M}^\star\right\|_{\mathrm{F}}$$

Main results: general case

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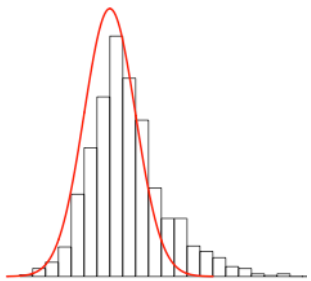
sample complexity bound $O(nr^2 \log^3 n)$ is suboptimal in r!

Inference and uncertainty quantification

— *inspired by Zhang, Zhang '11, van de Geer et al. '13, Javanmard, Montanari '13*



convex optimizer

M^{CVX}

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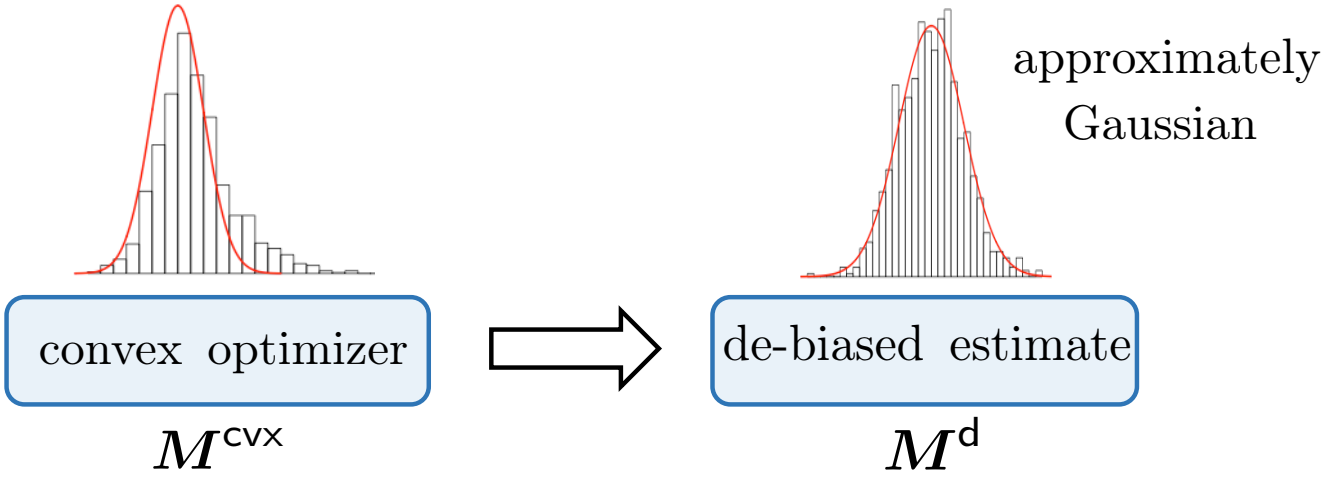

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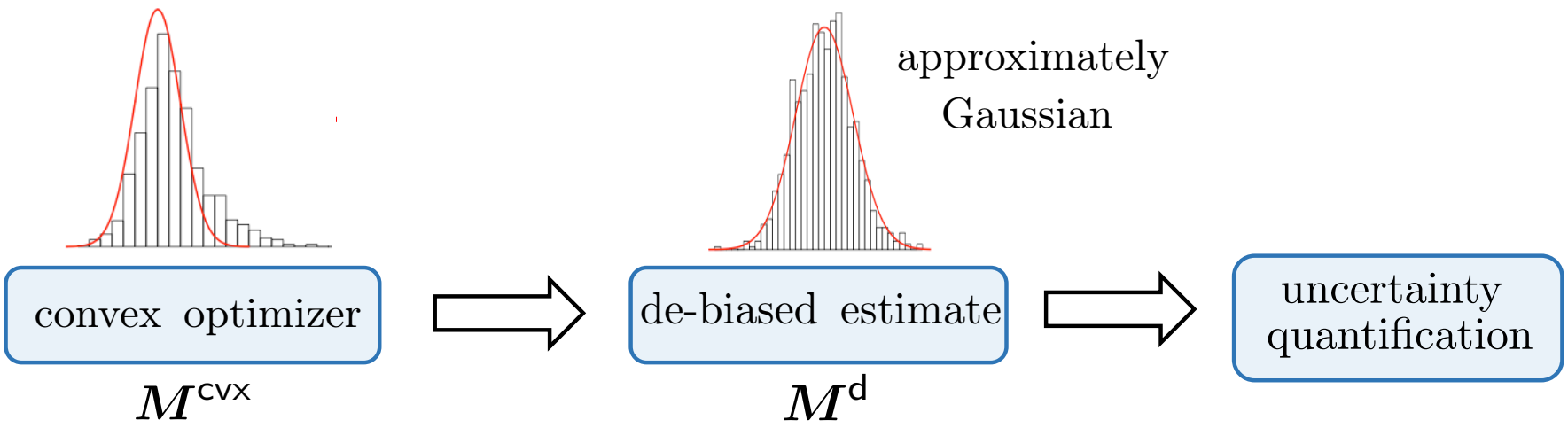

De-biasing convex estimate

$$\boldsymbol{M}^{\mathsf{cvx}} \quad \overset{\text{de-biasing}}{\longrightarrow} \quad \underbrace{\boldsymbol{M}^{\mathsf{cvx}} + \frac{1}{p}\mathcal{P}_{\Omega}(\boldsymbol{M}^{\star} + \boldsymbol{E} - \boldsymbol{M}^{\mathsf{cvx}})}_{\text{(nearly) unbiased estimate of } \boldsymbol{M}^{\star}}$$

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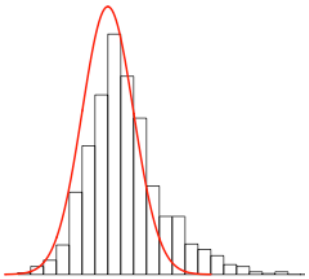
- **issue:** high-rank after de-biasing; statistical accuracy suffers

De-biasing convex estimate

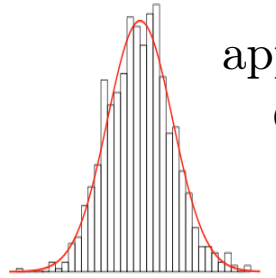
$$M^{\text{cvx}} \overset{\text{de-biasing}}{\longrightarrow} \underbrace{\text{proj}_{\text{rank-}r}\Big(M^{\text{cvx}} + \tfrac{1}{p}\mathcal{P}_{\Omega}(M^{\star} + \boldsymbol{E} - M^{\text{cvx}}) \Big)}_{\text{1 iteration of singular value projection (Jain, Meka, Dhillon '10)}} =: M^{\text{d}}$$

- **issue:** high-rank after de-biasing; statistical accuracy suffers
- **solution:** low-rank projection



convex optimizer

M^{cvx}



approximately Gaussian

de-biased estimate

M^{d}

uncertainty quantification

Distributional guarantees for low-rank factors

- **random sampling:** each $(i,j) \in \Omega$ with prob. $p \gtrsim \frac{\log^3 n}{n}$
- **random noise:** i.i.d. $\mathcal{N}(0, \sigma^2)$ (not too large)
- true matrix $\boldsymbol{M}^\star \in \mathbb{R}^{n \times n}$: $r = O(1)$, incoherent, well-conditioned
- regularization parameter: $\lambda \asymp \sigma\sqrt{np}$

$$\boldsymbol{X}^{\mathsf{d}}\boldsymbol{Y}^{\mathsf{d}\top} \leftarrow \underbrace{\text{balanced}}_{\boldsymbol{X}^{\mathsf{d}\top}\boldsymbol{X}^{\mathsf{d}} = \boldsymbol{Y}^{\mathsf{d}\top}\boldsymbol{Y}^{\mathsf{d}}} \text{ rank-}r \text{ decomp. of } \boldsymbol{M}^{\mathsf{d}}$$

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Theorem 3 (Chen, Fan, Ma, Yan '19)

With high prob., there exists global rotation matrix $\boldsymbol{H} \in \mathbb{R}^{r\times r}$ *s.t.*

$$\boldsymbol{X}^{\mathrm{d}}\boldsymbol{H} - \boldsymbol{X}^{\star} \approx \boldsymbol{Z}^{X}, \qquad \boldsymbol{Z}^{X}_{i,\cdot} \overset{\text{ind.}}{\sim} \mathcal{N}(\boldsymbol{0}, \tfrac{\sigma^2}{p}(\boldsymbol{Y}^{\star\top}\boldsymbol{Y}^{\star})^{-1})$$

$$\boldsymbol{Y}^{\mathrm{d}}\boldsymbol{H} - \boldsymbol{Y}^{\star} \approx \boldsymbol{Z}^{Y}, \qquad \boldsymbol{Z}^{Y}_{i,\cdot} \overset{\text{ind.}}{\sim} \mathcal{N}(\boldsymbol{0}, \tfrac{\sigma^2}{p}(\boldsymbol{X}^{\star\top}\boldsymbol{X}^{\star})^{-1})$$

Implications

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- estimation errors for different rows of $\boldsymbol{X}^{\star}$ are nearly independent

$$\boldsymbol{X}^{\mathrm{d}}_{i,\cdot}\boldsymbol{H} - \boldsymbol{X}^{\star}_{i,\cdot} \quad \text{nearly ind. of} \quad \boldsymbol{X}^{\mathrm{d}}_{j,\cdot}\boldsymbol{H} - \boldsymbol{X}^{\star}_{j,\cdot}$$

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$$\boldsymbol{X}^{\mathsf{d}}_{i,\cdot}\boldsymbol{H} - \boldsymbol{X}^{\star}_{i,\cdot} \sim \mathcal{N}(\boldsymbol{0}, \tfrac{\sigma^2}{p}(\boldsymbol{Y}^{\star\top}\boldsymbol{Y}^{\star})^{-1}) + \text{negligible term}$$

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$$M^{\mathsf{d}}_{i,j} - M^{\star}_{i,j} \;\sim\; \mathcal{N}\big(0, v^{\star}_{i,j}\big) + \text{negligible term}$$

where $v^{\star}_{i,j} \triangleq \frac{\sigma^2}{p}\Big\{\boldsymbol{X}^{\star}_{i,\cdot}(\boldsymbol{X}^{\star\top}\boldsymbol{X}^{\star})^{-1}\boldsymbol{X}^{\star\top}_{i,\cdot} + \boldsymbol{Y}^{\star}_{j,\cdot}(\boldsymbol{Y}^{\star\top}\boldsymbol{Y}^{\star})^{-1}\boldsymbol{Y}^{\star\top}_{j,\cdot}\Big\}$

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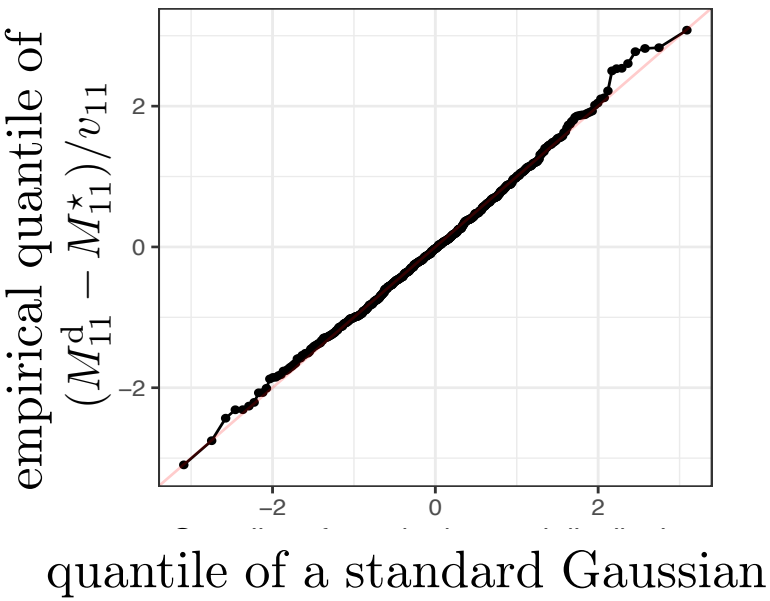
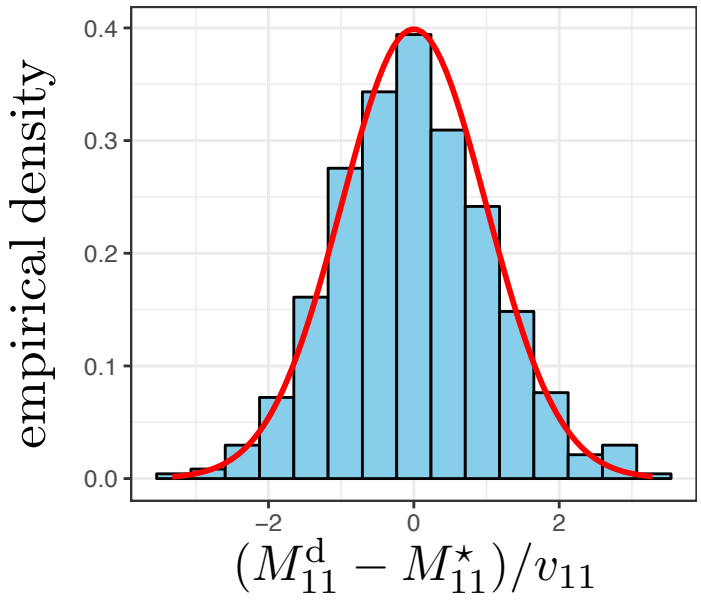
Numerical experiments





$$n = 1000,\ p = 0.2,\ r = 5,\ \|\boldsymbol{M}^{\star}\| = 1,\ \kappa = 1,\ \sigma = 10^{-3}$$

convex

nonconvex

convex

nonconvex

$$\Downarrow$$

$$\text{inference}\left(\text{convex}\right) \approx \text{inference}\left(\text{nonconvex}\right)$$

Same inference procedures work for both cvx & noncvx estimates!

A bit of intuition

Consider rank-1 PSD case $\boldsymbol{M}^\star = \boldsymbol{x}^\star \boldsymbol{x}^{\star\top}$, $p = 1$ (no missing data)

$$\text{minimize}_{\boldsymbol{x}} \qquad \frac{1}{2}\|\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star \boldsymbol{x}^{\star\top} - \boldsymbol{E}\|_{\mathrm{F}}^2 + \lambda \|\boldsymbol{x}\|_2^2$$

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- first-order optimality condition

$$(\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star \boldsymbol{x}^{\star\top} - \boldsymbol{E})\boldsymbol{x} + \lambda \boldsymbol{x} = \boldsymbol{0}$$

A bit of intuition

$$(\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top} - \boldsymbol{E})\boldsymbol{x} \underbrace{+\lambda\boldsymbol{x}}_{\text{causes bias}} = \boldsymbol{0}$$

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$$\Updownarrow$$

$$(\boldsymbol{x}^{\mathrm{d}}\boldsymbol{x}^{\mathrm{d}\top} - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top} - \boldsymbol{E})\boldsymbol{x}^{\mathrm{d}} = \boldsymbol{0}, \qquad \boldsymbol{x}^{\mathrm{d}} = \sqrt{\frac{\lambda+\|\boldsymbol{x}\|_2^2}{\|\boldsymbol{x}\|_2^2}}\,\boldsymbol{x}$$

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$$\Updownarrow$$

$$\boldsymbol{x}^{\mathrm{d}} - \boldsymbol{x}^\star = \underbrace{\frac{1}{\|\boldsymbol{x}^{\mathrm{d}}\|_2^2}\boldsymbol{E}\boldsymbol{x}^{\mathrm{d}}}_{\text{nearly Gaussian}} + \underbrace{\frac{\left(\boldsymbol{x}^\star - \boldsymbol{x}^{\mathrm{d}}\right)^\top \boldsymbol{x}^{\mathrm{d}}}{\|\boldsymbol{x}^{\mathrm{d}}\|_2^2}\boldsymbol{x}^\star}_{\text{hopefully small}}$$

An alternative de-biased estimator

— inspired by Zhang, Zhang '11, van de Geer et al. '13, Javanmard, Montanari '13

$$\boldsymbol{M}^{\mathsf{CVX}} \triangleq \underset{\boldsymbol{Z}\in\mathbb{R}^{n\times n}}{\arg\min} \ \left\|\mathcal{P}_{\Omega}(\boldsymbol{Z}-\boldsymbol{M})\right\|_{\mathrm{F}}^{2} + \lambda\|\boldsymbol{Z}\|_{*}$$

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$$\Downarrow$$

$$\widehat{\boldsymbol{M}}^{\mathrm{d}} = \boldsymbol{M}^{\mathsf{cvx}} + \text{linear-map}\Big(\mathcal{P}_{\Omega}\big(\boldsymbol{M}-\boldsymbol{M}^{\mathsf{cvx}}\big)\Big)$$

An alternative de-biased estimator

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$$\Downarrow$$

$$\widehat{\boldsymbol{M}}^{\mathrm{d}} = \boldsymbol{M}^{\mathsf{cvx}} + \tfrac{1}{p}\mathcal{P}_T\Big(\mathcal{P}_{\Omega}\big(\boldsymbol{M}-\boldsymbol{M}^{\mathsf{cvx}}\big)\Big)$$

— $\mathcal{P}_T$: projection onto T (tangent space at $\mathsf{proj}_{\mathsf{rank\text{-}r}}(\boldsymbol{M}^{\mathsf{cvx}})$)

$$\widehat{\boldsymbol{M}}^{\mathrm{d}} \approx \boldsymbol{M}^{\mathrm{d}} \ !$$

Back to estimation: de-biased estimator is optimal

Distributional theory in turn allows us to track estimation accuracy

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Distributional theory in turn allows us to track estimation accuracy

Theorem 4 (Chen, Fan, Ma, Yan '19)

$$\|\boldsymbol{M}^{\mathsf{d}} - \boldsymbol{M}^{\star}\|_{\mathrm{F}}^2 = \underbrace{\frac{(2+o(1))nr\sigma^2}{p}}_{\textit{Cramer-Rao lower bound}} \qquad \textit{with high prob.}$$

Back to estimation: de-biased estimator is optimal

Distributional theory in turn allows us to track estimation accuracy

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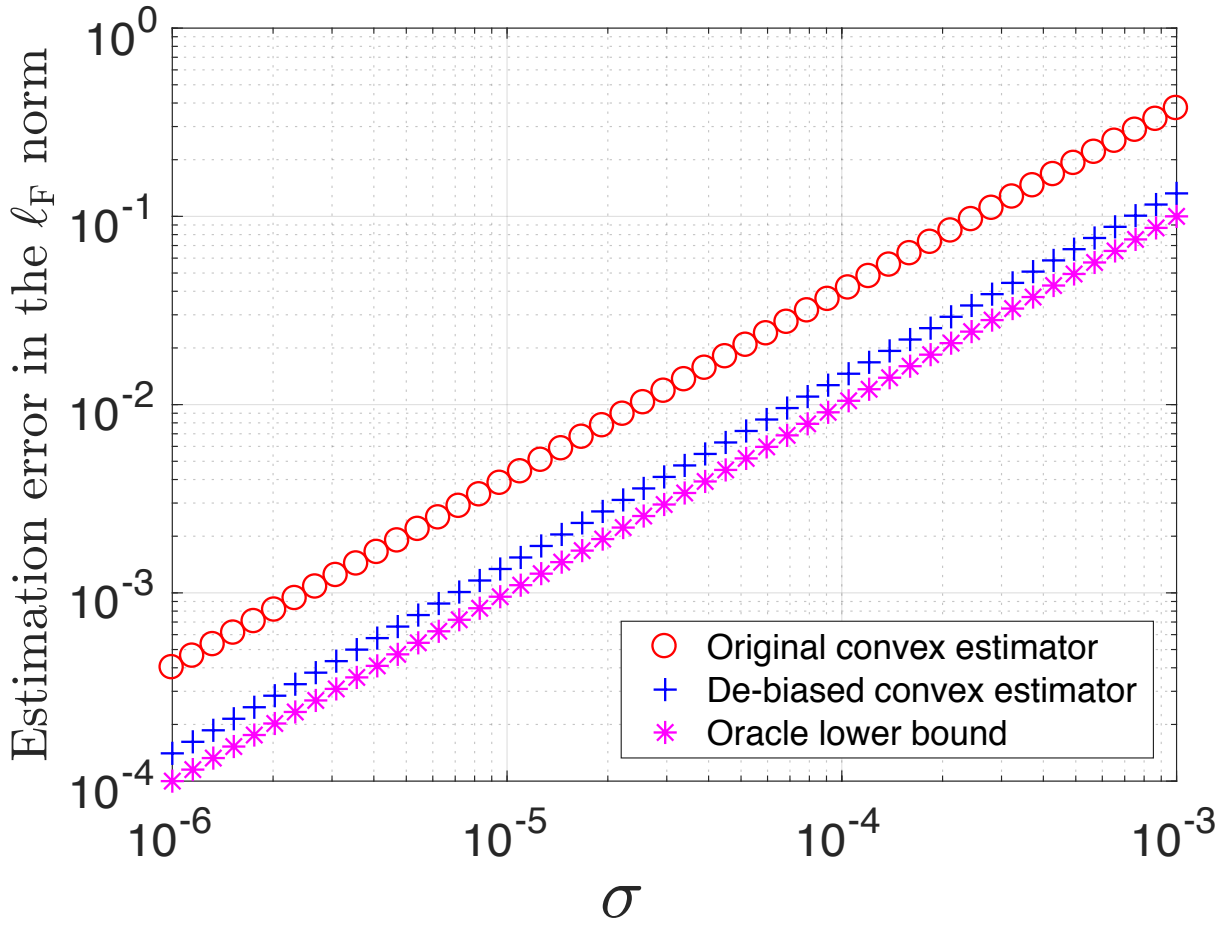
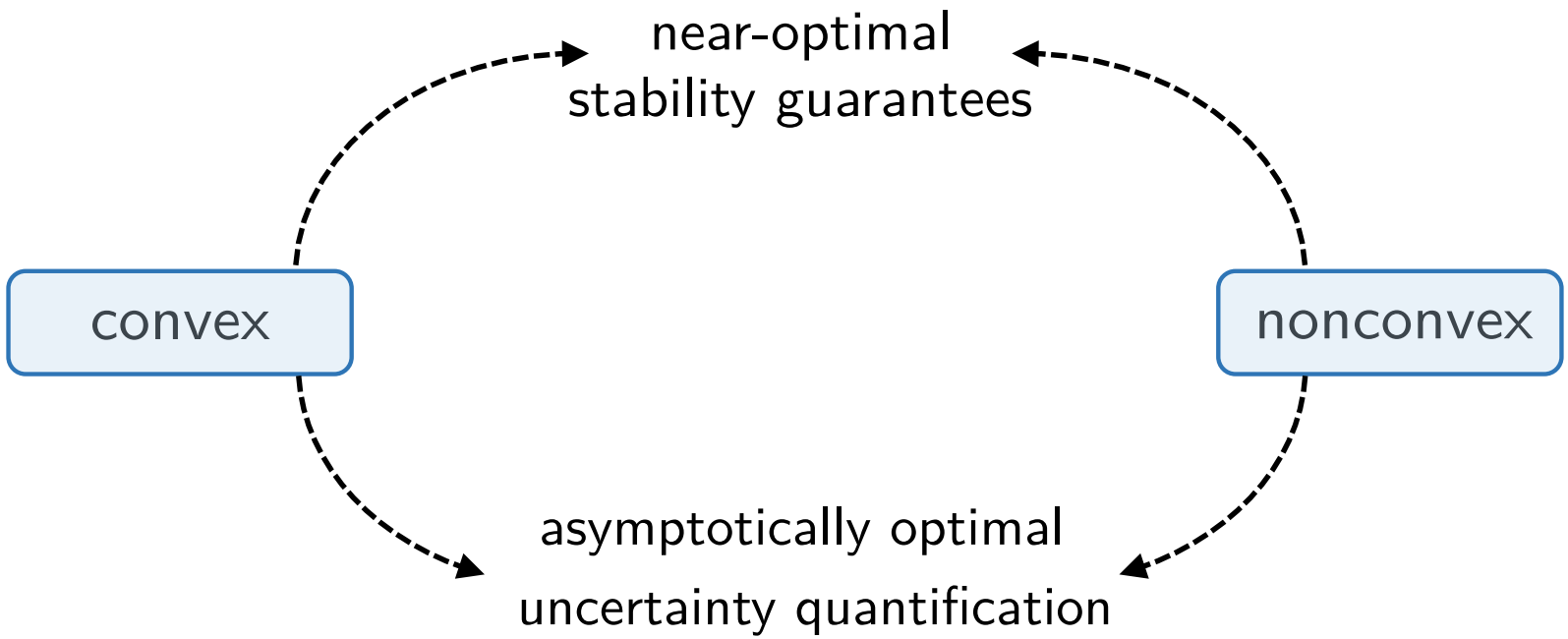
- precise characterization of estimation accuracy
- achieves full statistical efficiency (including pre-constant)

Numerical evidence ($r = 5$, $p = 0.2$, $n = 1000$)

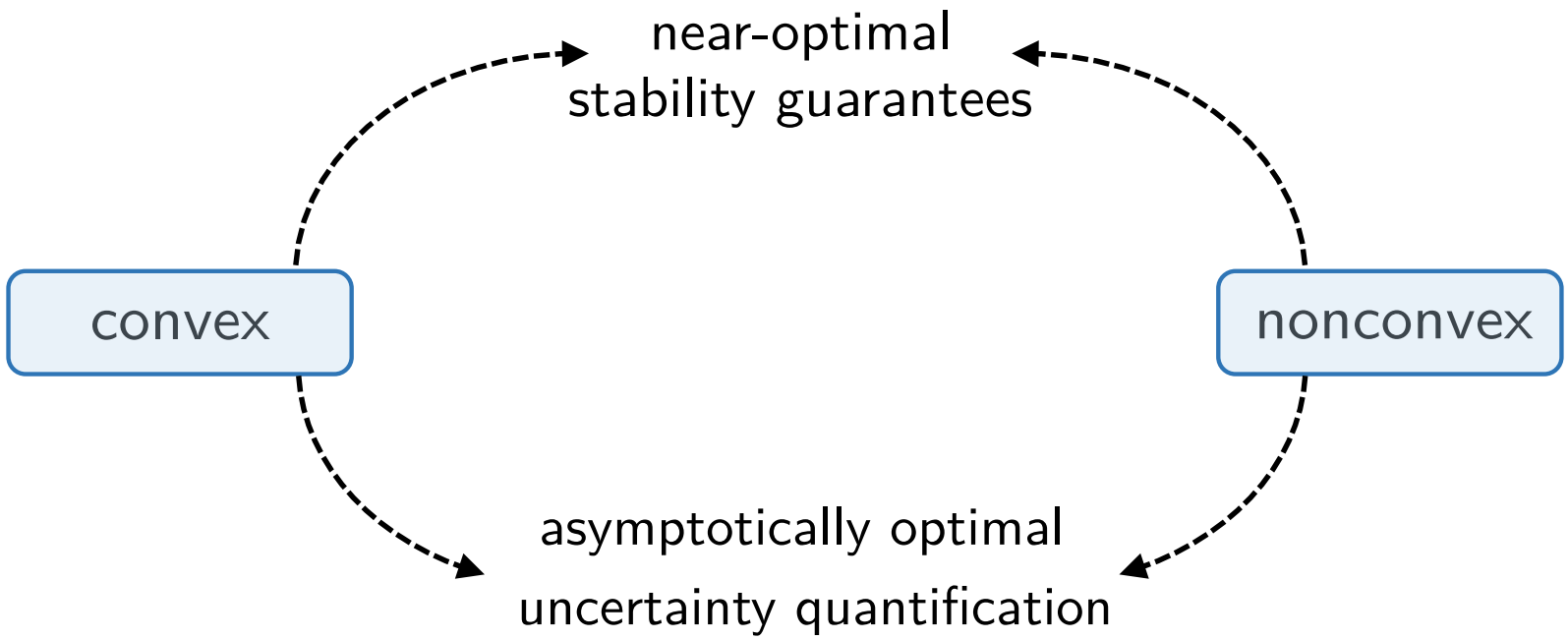


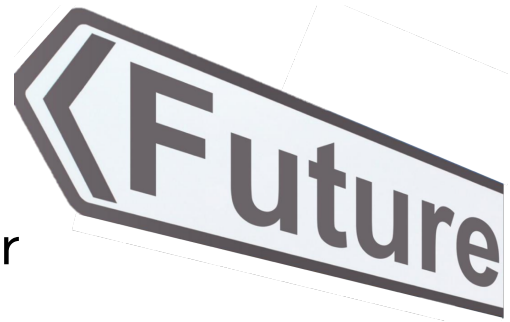
Euclidean estimation error vs. noise standard deviation σ

Concluding remarks

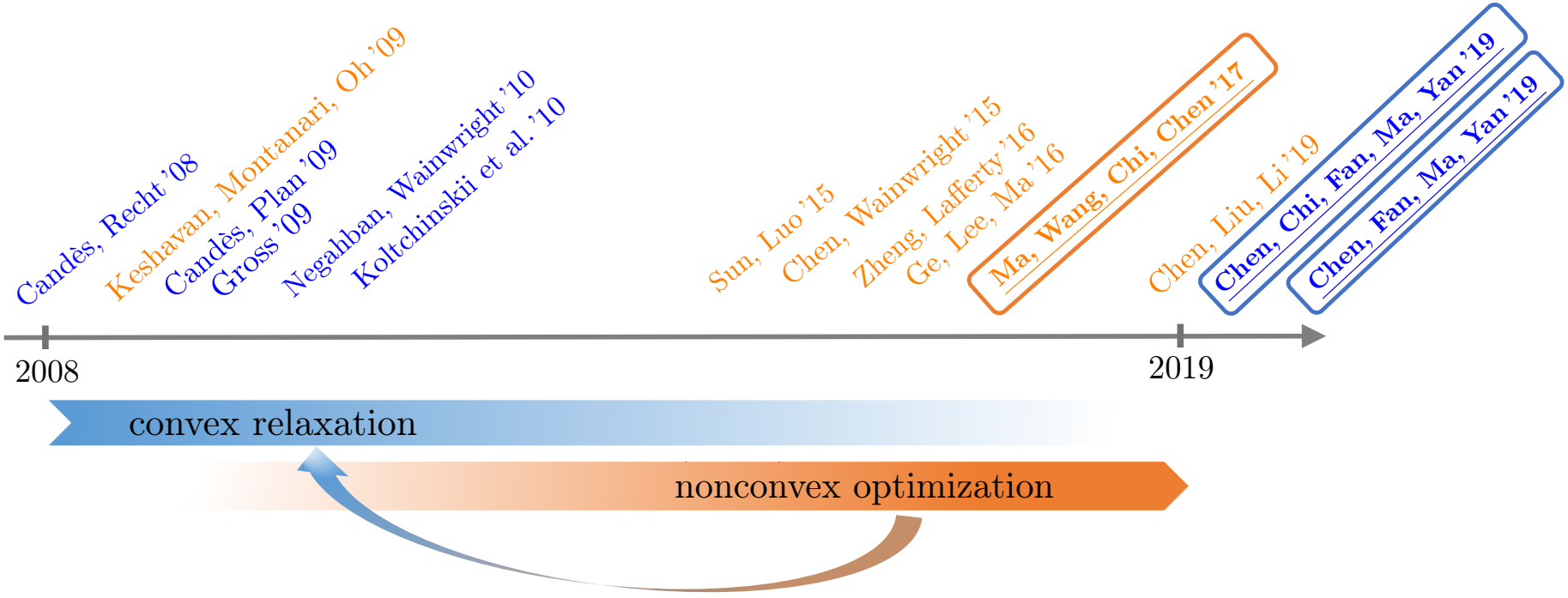


Concluding remarks





- improve dependency on rank & cond. number
- what if $\boldsymbol{M}^\star$ is only approximately low-rank?
- more general sampling patterns / graphs



"Noisy matrix completion: understanding statistical guarantees for convex relaxation via nonconvex optimization," Y. Chen, Y. Chi, J. Fan, C. Ma, Y. Yan, 2019

"Inference and uncertainty quantification for noisy matrix completion," Y. Chen, J. Fan, C. Ma, Y. Yan, 2019

Backup slides — a little analysis:
connection between convex and nonconvex solutions

Link between convex and nonconvex optimizers

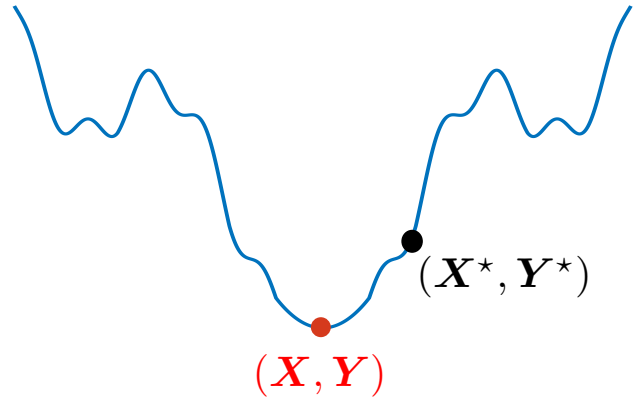
$(\boldsymbol{X}, \boldsymbol{Y})$ is nonconvex optimizer

Link between convex and nonconvex optimizers

$(\boldsymbol{X}, \boldsymbol{Y})$ is nonconvex optimizer $\overset{?}{\Longrightarrow}$ $\boldsymbol{X}\boldsymbol{Y}^\top$ is convex solution

Link between convex and nonconvex optimizers

- $(\boldsymbol{X}, \boldsymbol{Y})$ is close to truth (in $\ell_{2,\infty}$ sense)
- a little condition on noise size



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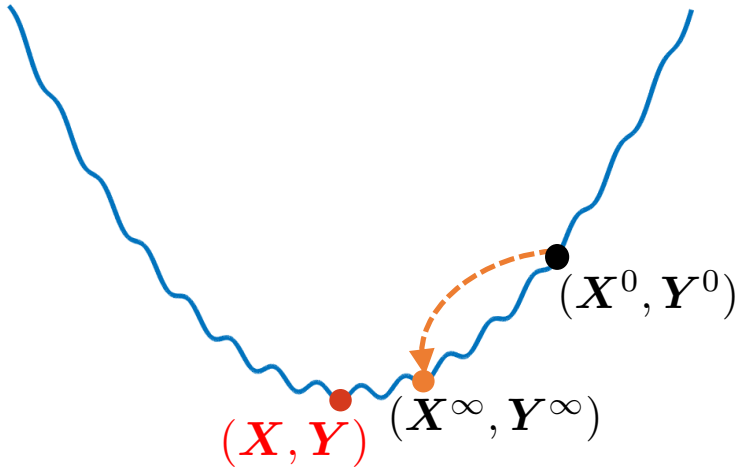
Approximate nonconvex optimizers



Issue: we do NOT know properties of nonconvex optimizers

- It is unclear whether nonconvex algorithms converge to optimizers (due to lack of strong convexity)

Approximate nonconvex optimizers

Strategy: resort to "$\underbrace{\text{approximate stationary points}}_{\nabla f(\boldsymbol{X},\boldsymbol{Y})\approx \mathbf{0}}$" instead

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starting from $(\boldsymbol{X}^0, \boldsymbol{Y}^0) =$ truth or spectral initialization:

$$\begin{aligned} \boldsymbol{X}^{t+1} &= \boldsymbol{X}^t - \eta \nabla_{\boldsymbol{X}} f(\boldsymbol{X}^t, \boldsymbol{Y}^t) \\ \boldsymbol{Y}^{t+1} &= \boldsymbol{Y}^t - \eta \nabla_{\boldsymbol{Y}} f(\boldsymbol{X}^t, \boldsymbol{Y}^t) \end{aligned} \qquad t = 0, 1, \cdots, T$$

Approximate nonconvex optimizers

Strategy: resort to "approximate stationary points" instead

$$\nabla f(\boldsymbol{X},\boldsymbol{Y}) \approx \boldsymbol{0}$$

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- when T is large: there exists point with very small gradient

$$\|\nabla f(\boldsymbol{X},\boldsymbol{Y})\|_{\mathrm{F}} \lesssim \frac{1}{\sqrt{\eta T}}$$

- hopefully not far from $(\boldsymbol{X}^\star, \boldsymbol{Y}^\star)$

Analyzing nonconvex GD: leave-one-out analysis

Leave out a small amount of information from data and run GD

Analyzing nonconvex GD: leave-one-out analysis

Leave out a small amount of information from data and run GD

- Stein '72
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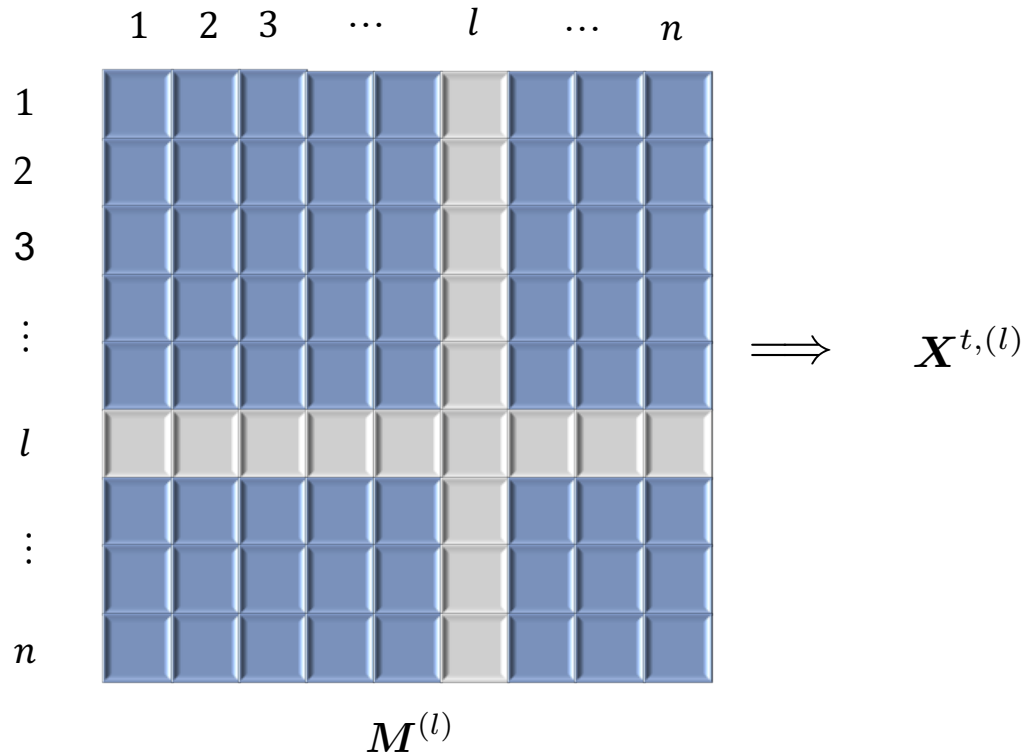
Analyzing nonconvex GD: leave-one-out analysis

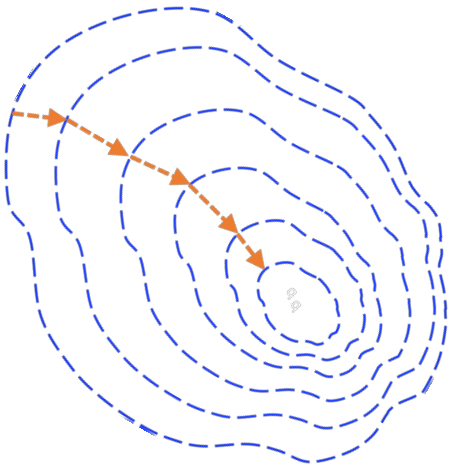
For each $1 \le l \le n$, introduce leave-one-out iterates $\boldsymbol{X}^{t,(l)}$ by replacing l^{th} row and column with true values



- exploit partial statistical independence
- exploit leave-one-out stability

Backup slides: gradient descent for nonconvex matrix completion

Gradient descent for nonconvex matrix completion



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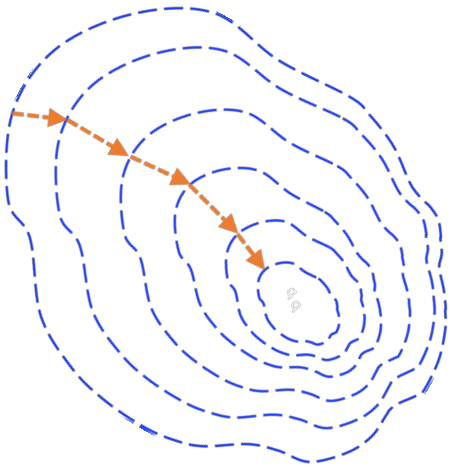
Prior works analyze regularized GD

- not guaranteed to return small-gradient solutions
- no $\ell_{2,\infty}$ error control

— *Keshavan et al. '09, Sun, Luo '15, Chen, Wainwright '15, Zheng, Lafferty '16*

Gradient descent for nonconvex matrix completion



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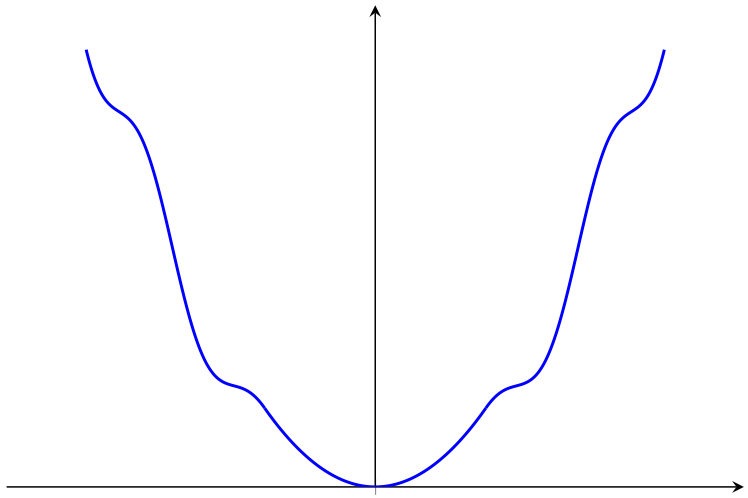
Our work and Chen et al. analyze vanilla GD

- regularization-free
- optimal $\ell_{2,\infty}$ error control

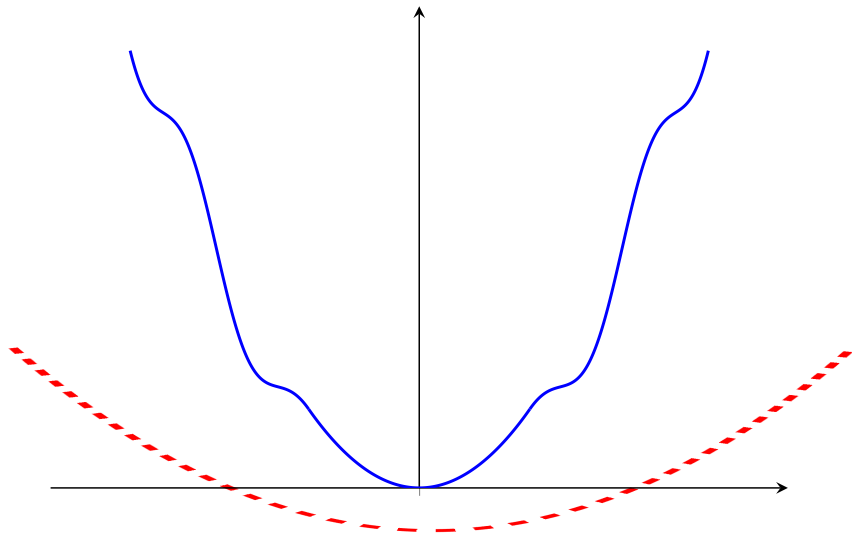
— Ma, Wang, Chi, Chen '17, Chen, Liu, Li '19

Gradient descent theory revisited



Two standard conditions that enable geometric convergence of GD

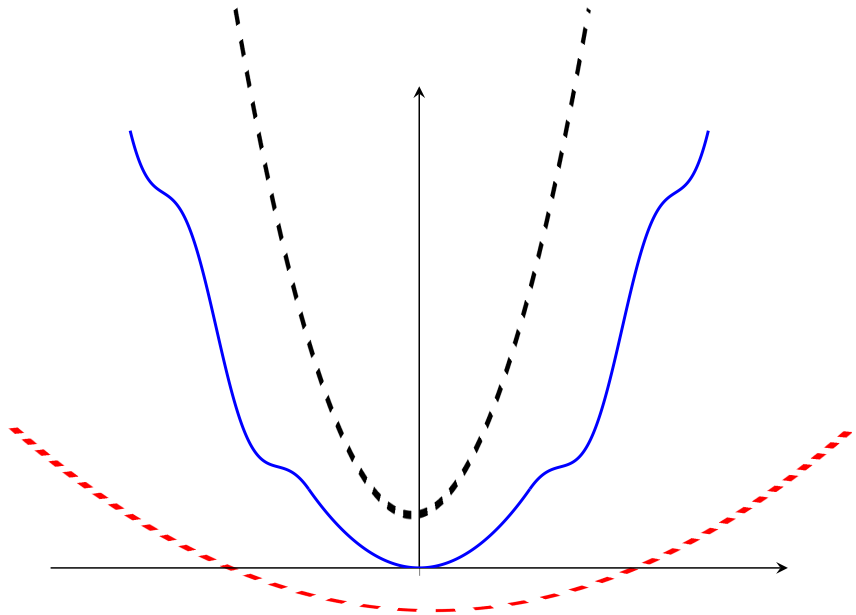
Gradient descent theory revisited



Two standard conditions that enable geometric convergence of GD

- (local) restricted strong convexity

Gradient descent theory revisited



Two standard conditions that enable geometric convergence of GD

- (local) restricted strong convexity
- (local) smoothness

Gradient descent theory revisited

f is said to be α-strongly convex and β-smooth if

$$\mathbf{0} \preceq \alpha \boldsymbol{I} \preceq \nabla^2 f(\boldsymbol{X}) \preceq \beta \boldsymbol{I}, \qquad \forall \boldsymbol{X}$$

ℓ_2 **error contraction:** GD with $\eta = 1/\beta$ obeys

$$\|\boldsymbol{X}^{t+1} - \boldsymbol{X}^\star\|_{\mathrm{F}} \leq \left(1 - \frac{\alpha}{\beta}\right) \|\boldsymbol{X}^t - \boldsymbol{X}^\star\|_{\mathrm{F}}$$

Incoherence region

Which region enjoys both restricted strong convexity and smoothness?

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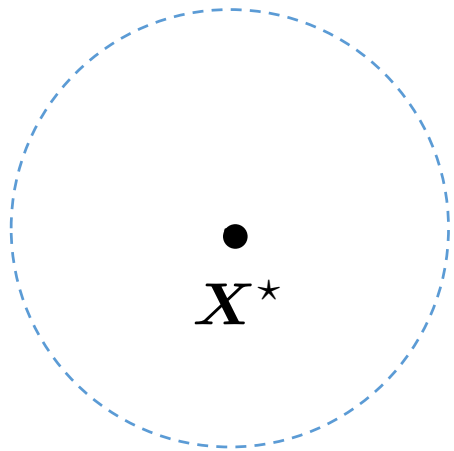
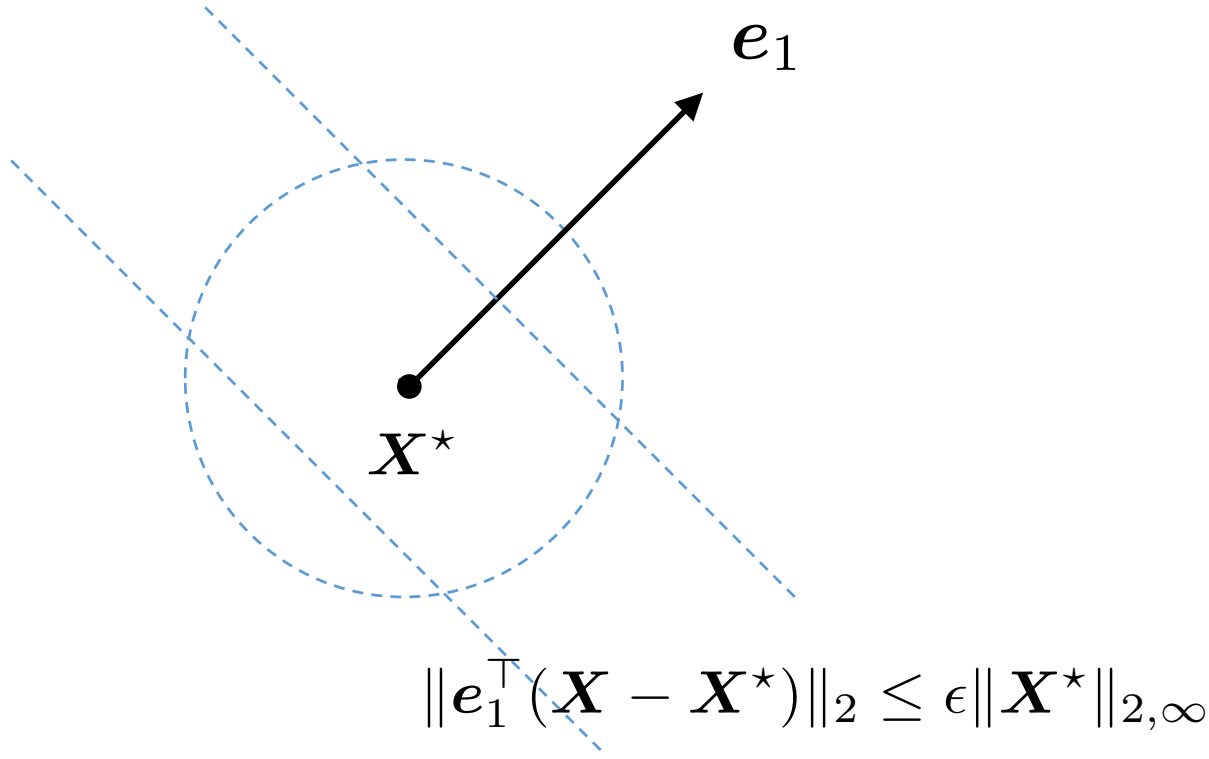


- $\boldsymbol{X}$ is not far away from $\boldsymbol{X}^\star$

Incoherence region

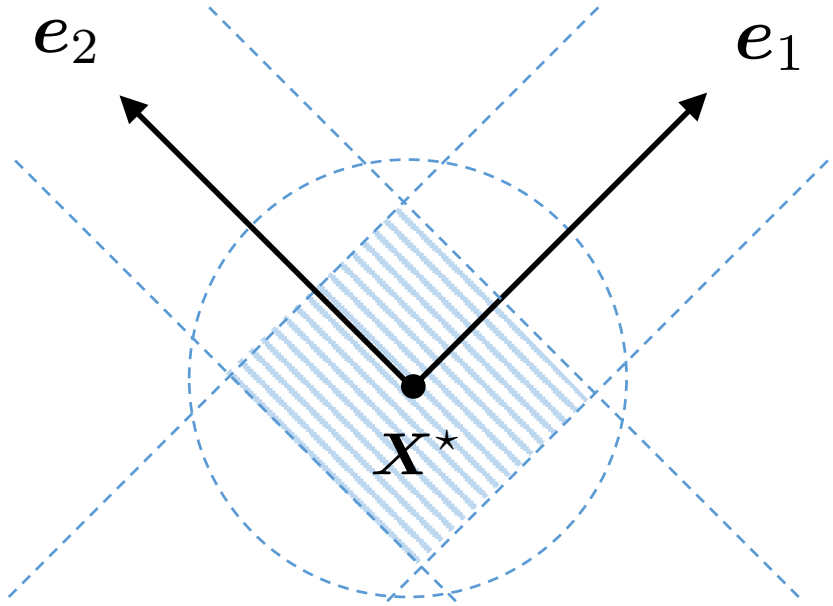
Which region enjoys both restricted strong convexity and smoothness?



- X is not far away from $X^\star$
- X is incoherent w.r.t. standard basis vectors (incoherence region)

Incoherence region

Which region enjoys both restricted strong convexity and smoothness?



$$\|e_2^\top (X - X^\star)\|_2 \le \epsilon \|X^\star\|_{2,\infty} \quad \|e_1^\top (X - X^\star)\|_2 \le \epsilon \|X^\star\|_{2,\infty}$$

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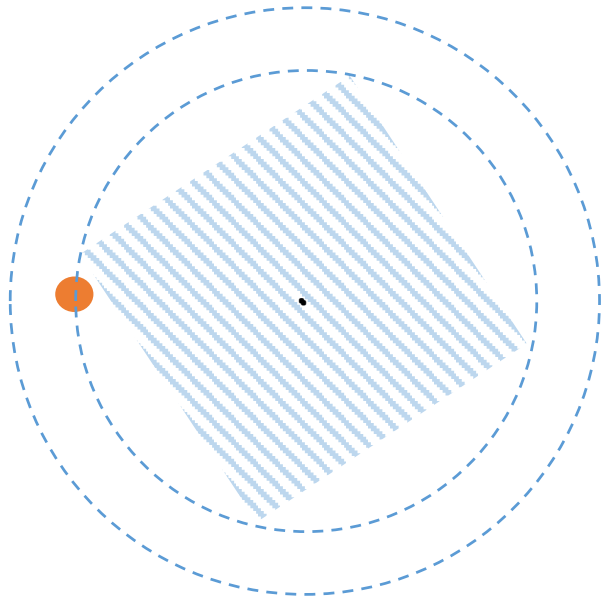
Inadequacy of generic gradient descent theory

region of local strong convexity + smoothness



- Generic optimization theory does NOT ensure GD stays in incoherence region

Inadequacy of generic gradient descent theory

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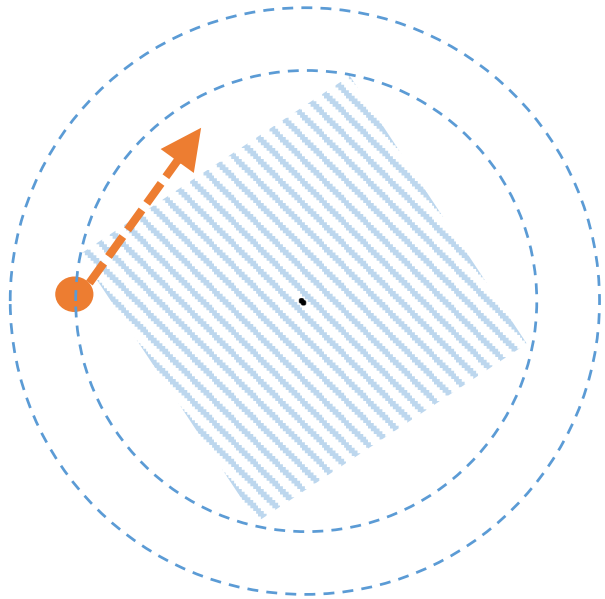

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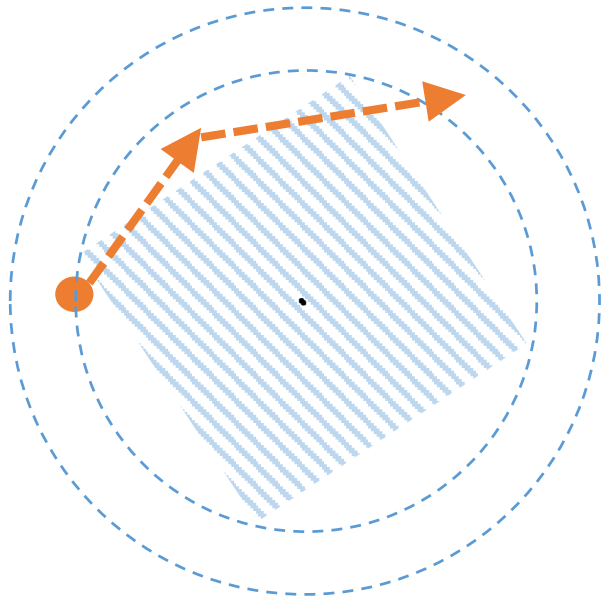

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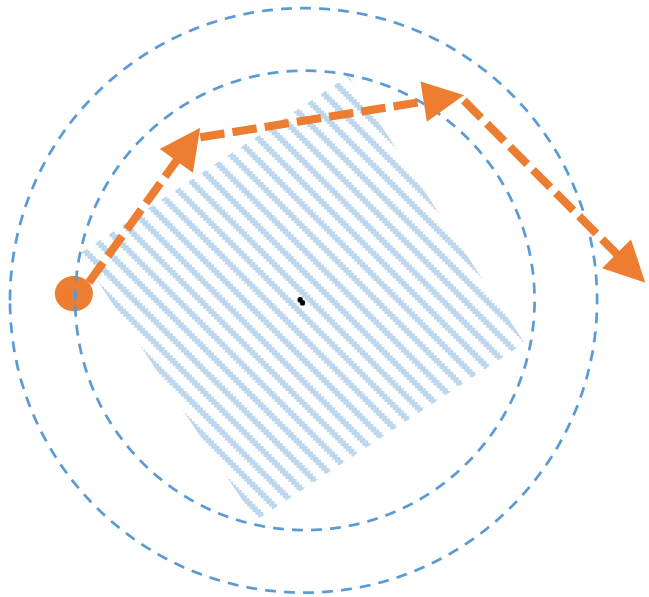

- Generic optimization theory does NOT ensure GD stays in incoherence region
- Calls for new analysis tools

Key proof idea: leave-one-out analysis

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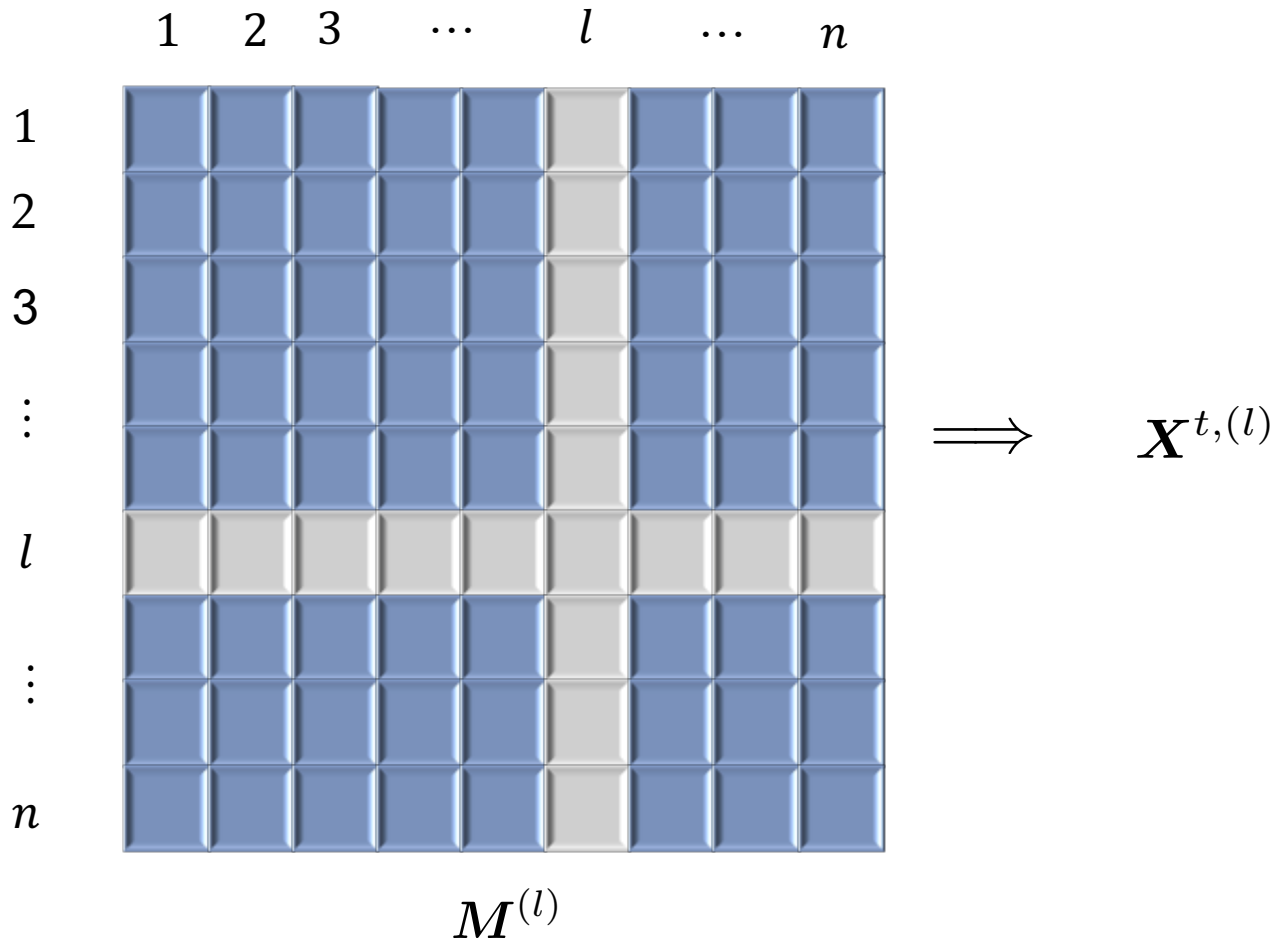

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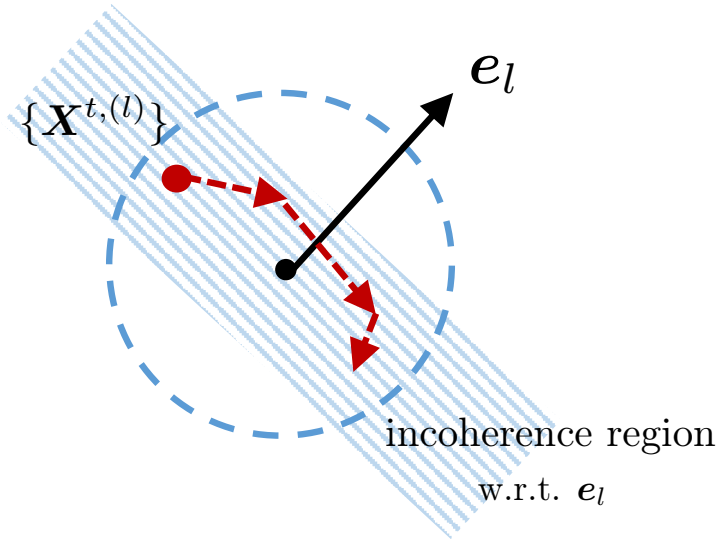

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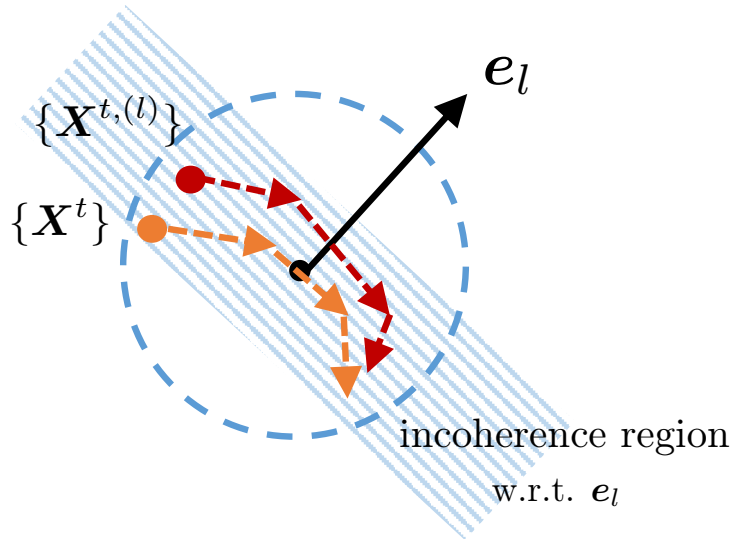

- Leave-one-out iterates $\{\boldsymbol{X}^{t,(l)}\}$ contains more information of l^{th} row of truth; indep. of randomness in l^{th} row
- Leave-one-out iterates $\{\boldsymbol{X}^{t,(l)}\}$ $\approx$ true iterates $\{\boldsymbol{X}^{t}\}$