ISIT 2013, Istanbul

July 9, 2013

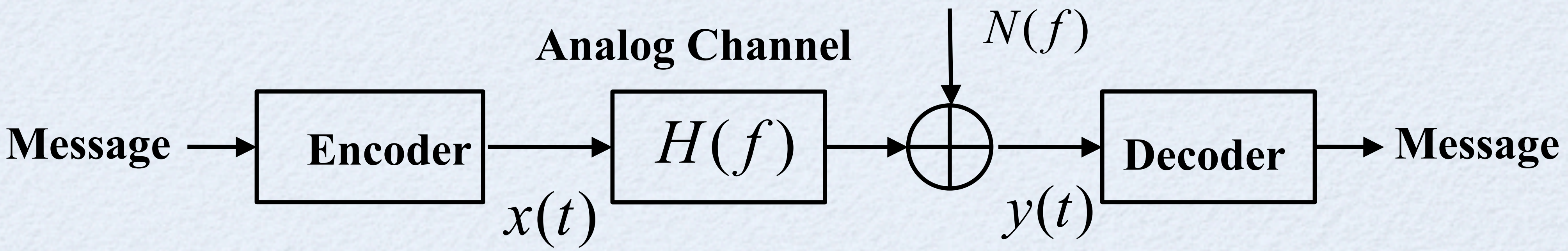
Minimax Universal Sampling for Compound Multiband Channels

Yuxin Chen, Andrea Goldsmith, Yonina Eldar

Stanford University ***Technion***

Capacity of Undersampled Channels

- **Point-to-point channels**



C. E. Shannon

Issue: *wideband systems preclude Nyquist-rate sampling!*

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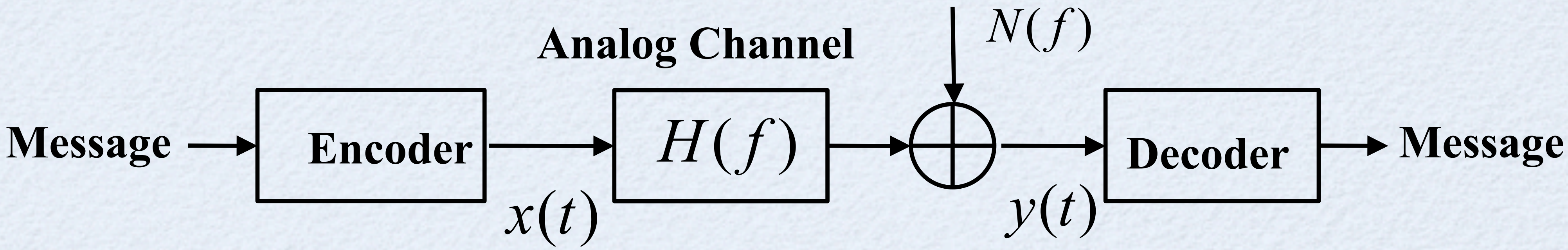


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H. Nyquist

- **Sub-Nyquist sampling well explored in Signal Processing**
  - *Landau-rate sampling, compressed sensing, etc.*
  - *Objective metric: MSE*

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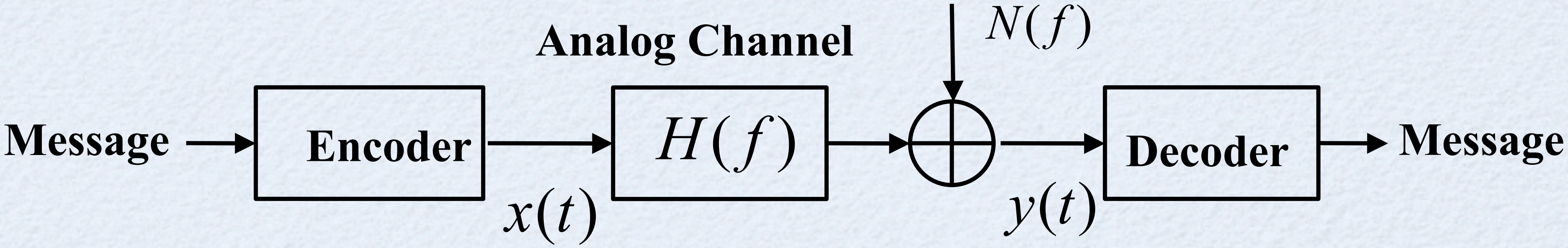

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- **Sub-Nyquist sampling well explored in Signal Processing**
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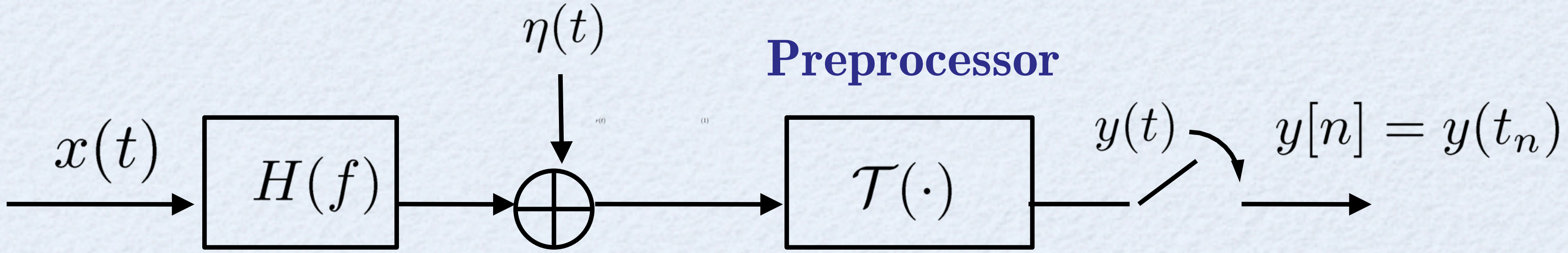
  - ***Objective metric: MSE***

- ***Question: which sub-Nyquist samplers are optimal in terms of CAPACITY?***

Prior work: Channel-specific Samplers

- **Consider linear time-invariant sub-sampled channels**



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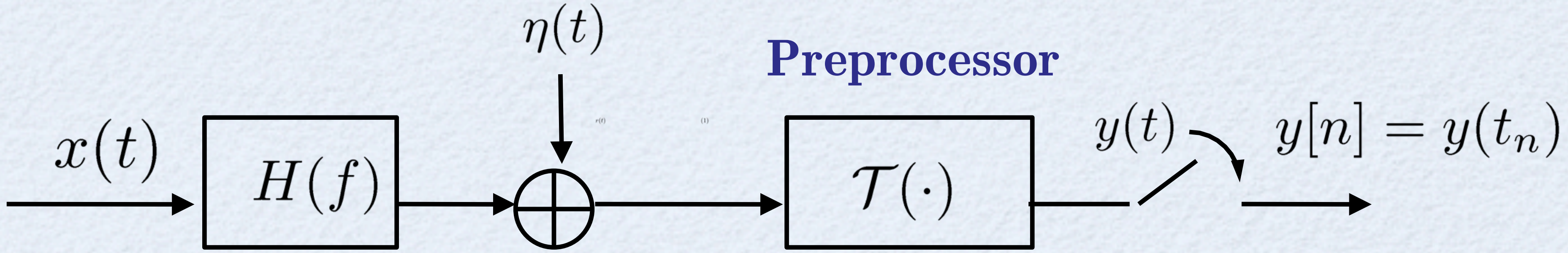
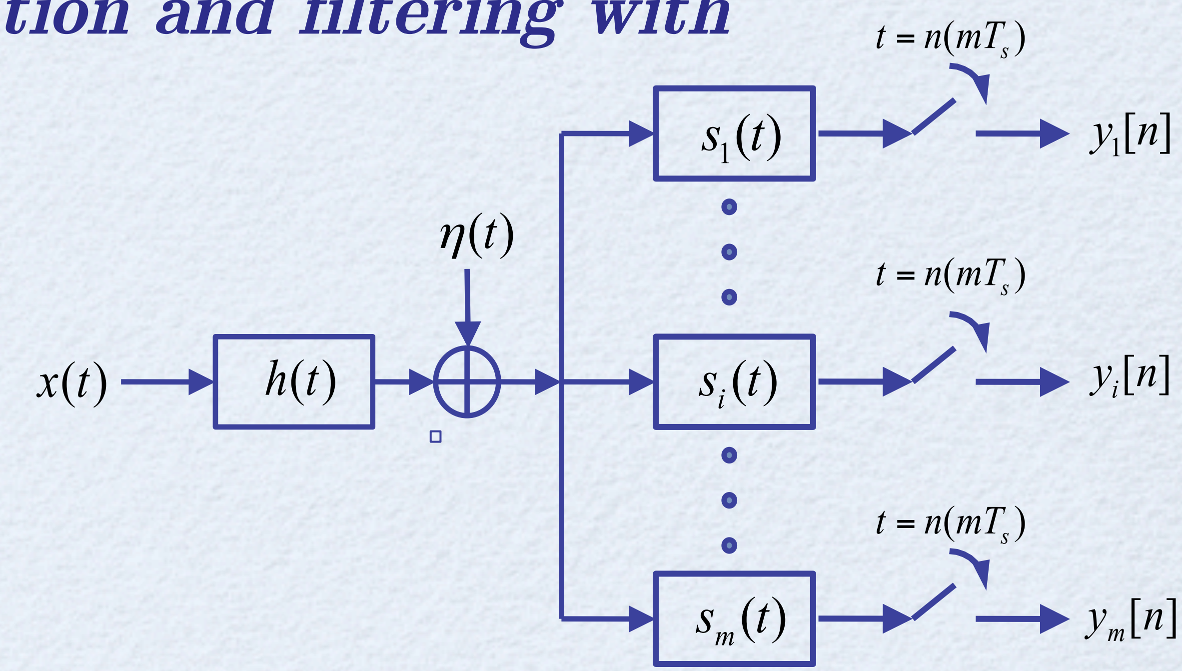


- The channel-optimized sampler ***(optimized for a single channel)***
  - *(1) a filter bank followed by uniform sampling*
  - *(2) a single branch of and modulation and filtering with uniform sampling*



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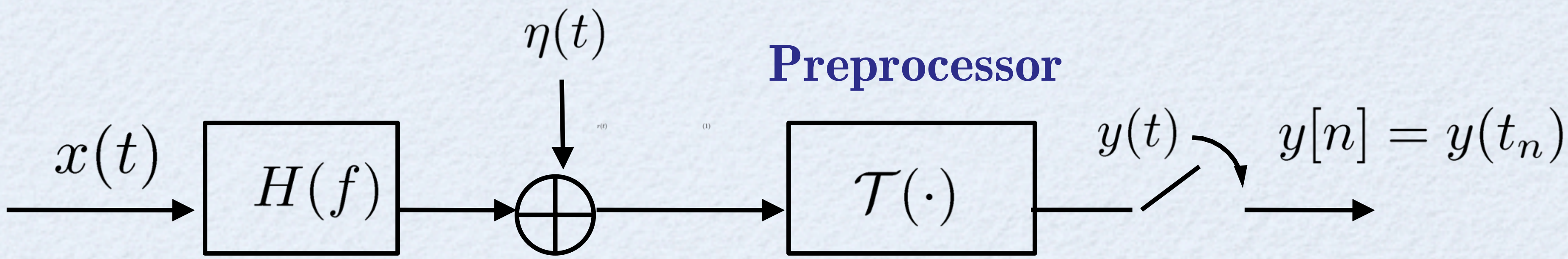

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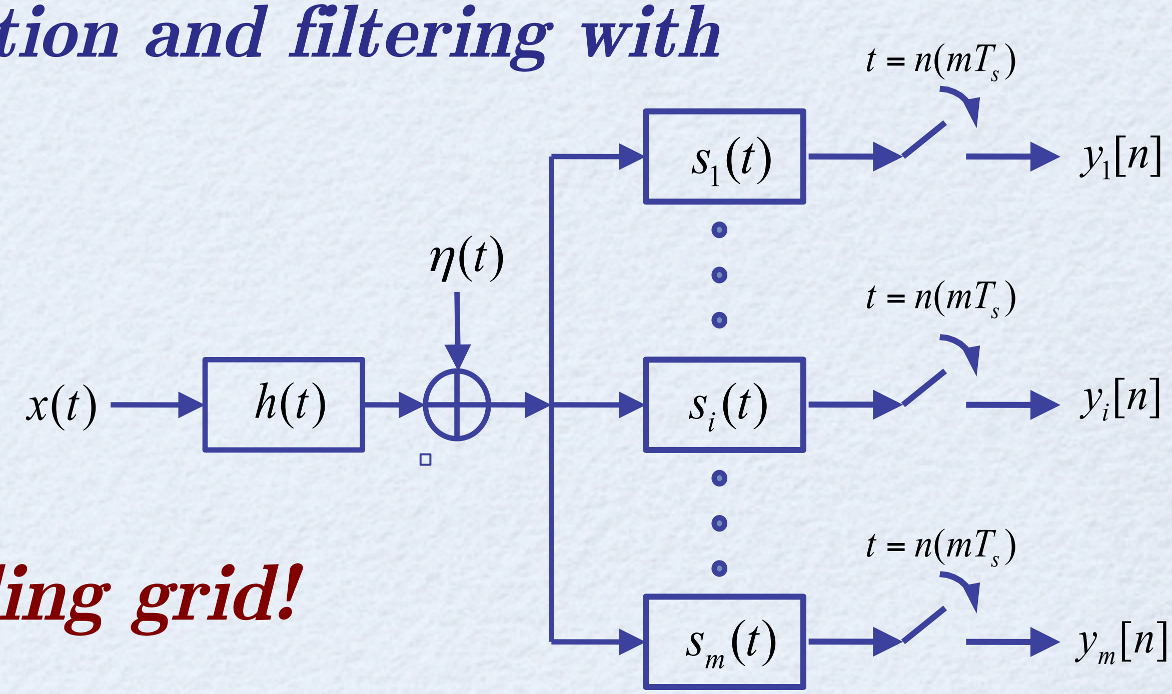


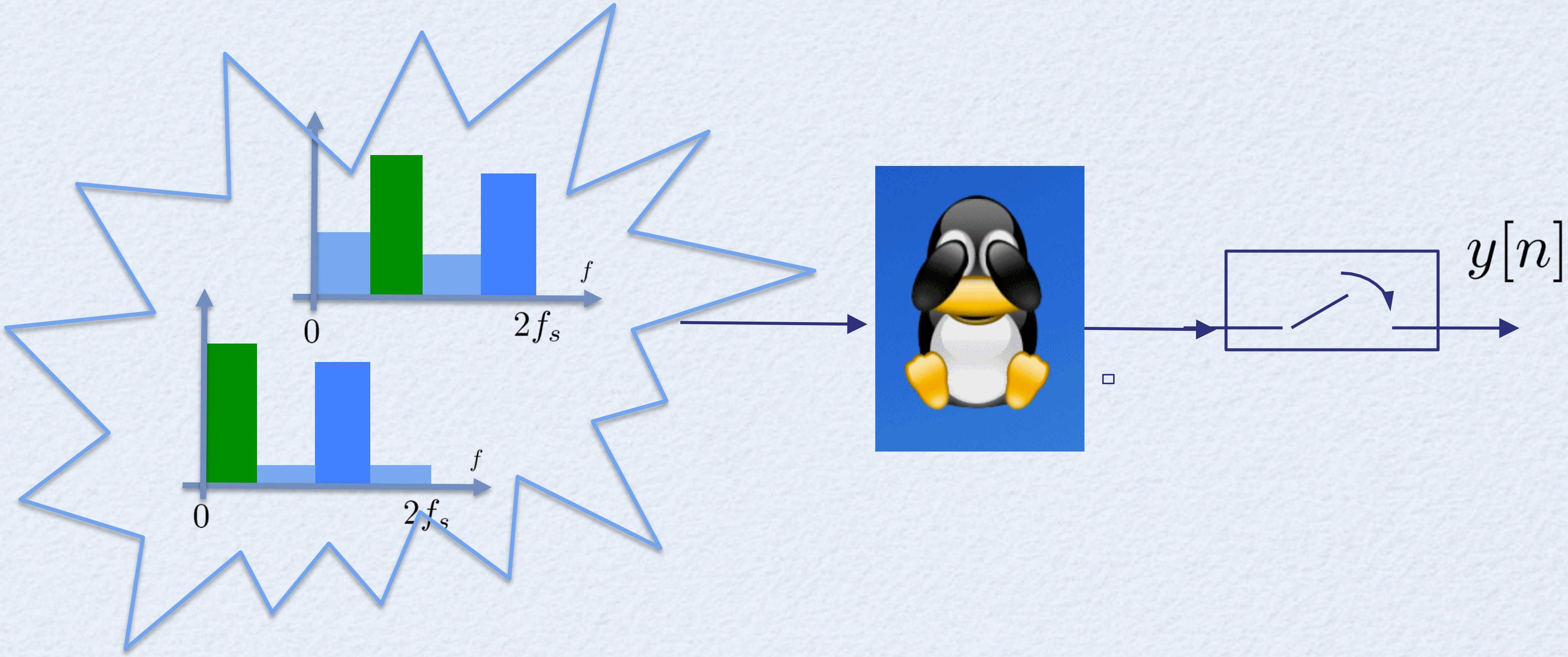
- ***Suppresses Aliasing***

- ***No need to use non-uniform sampling grid!***

Universal Sampling for Compound Channels

The channel-optimized sampler suppresses aliasing

- **What if there are a collection of channel realizations?**



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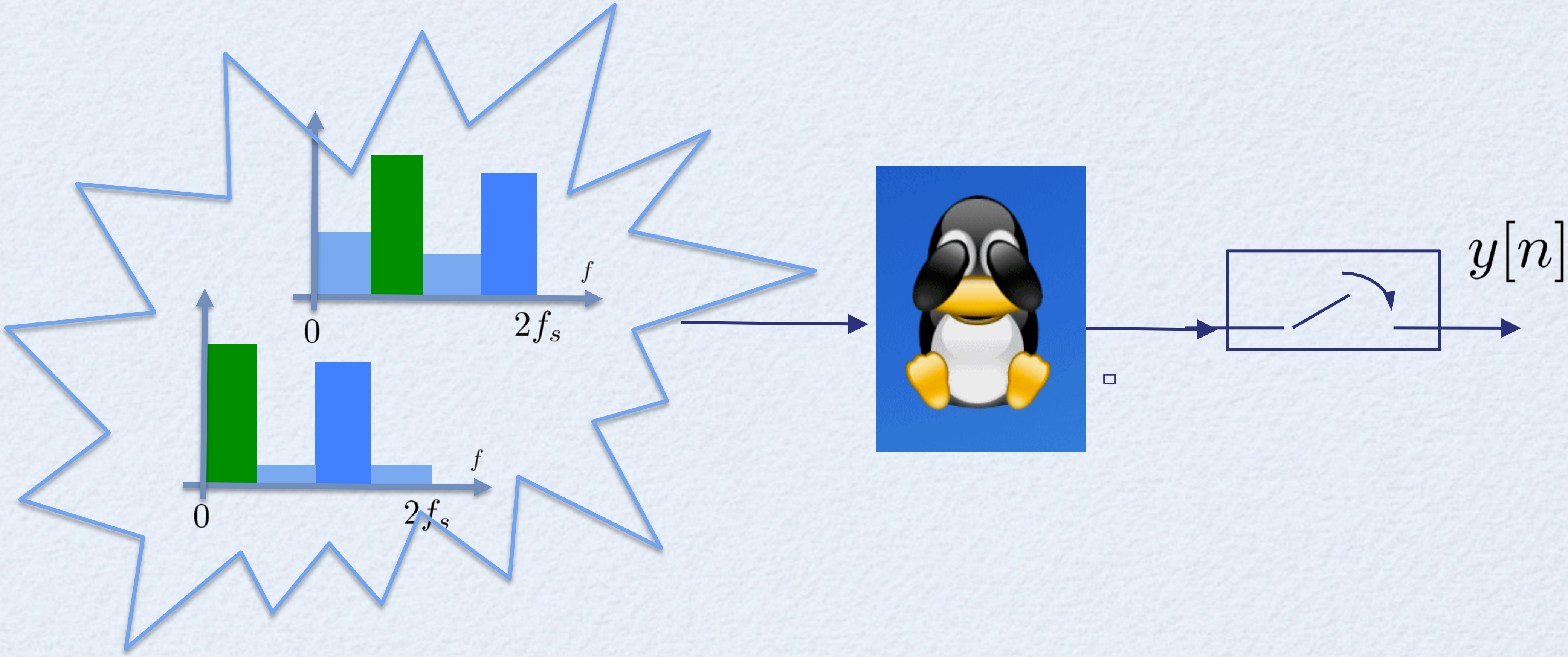


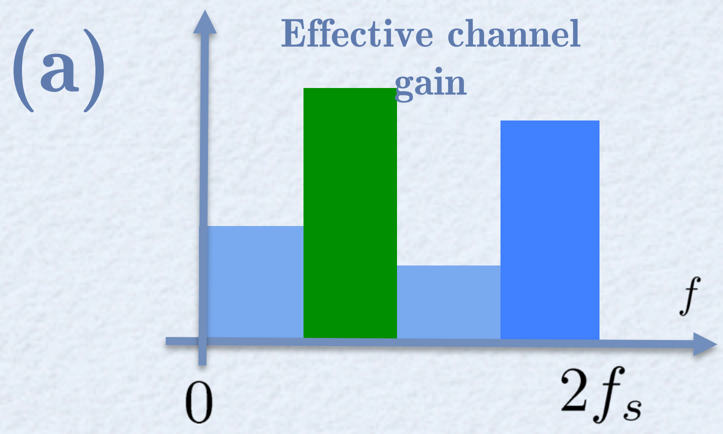
- **Universal** (***channel-blind***) Sampling

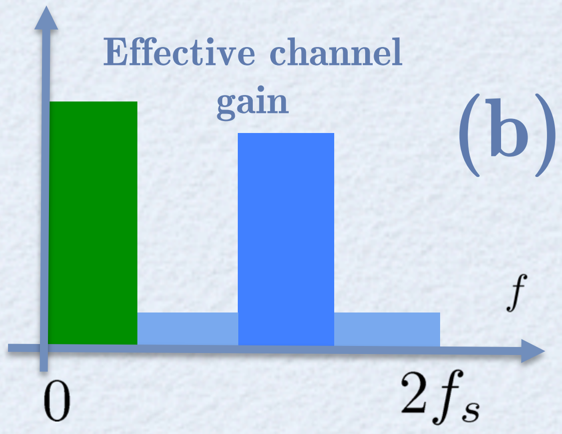
---- A sampler is typically integrated into the hardware

---- Need to operate *independently* of instantaneous realization

Sub-optimality of Channel-optimized Samplers

Consider 2 possible channel realizations ………





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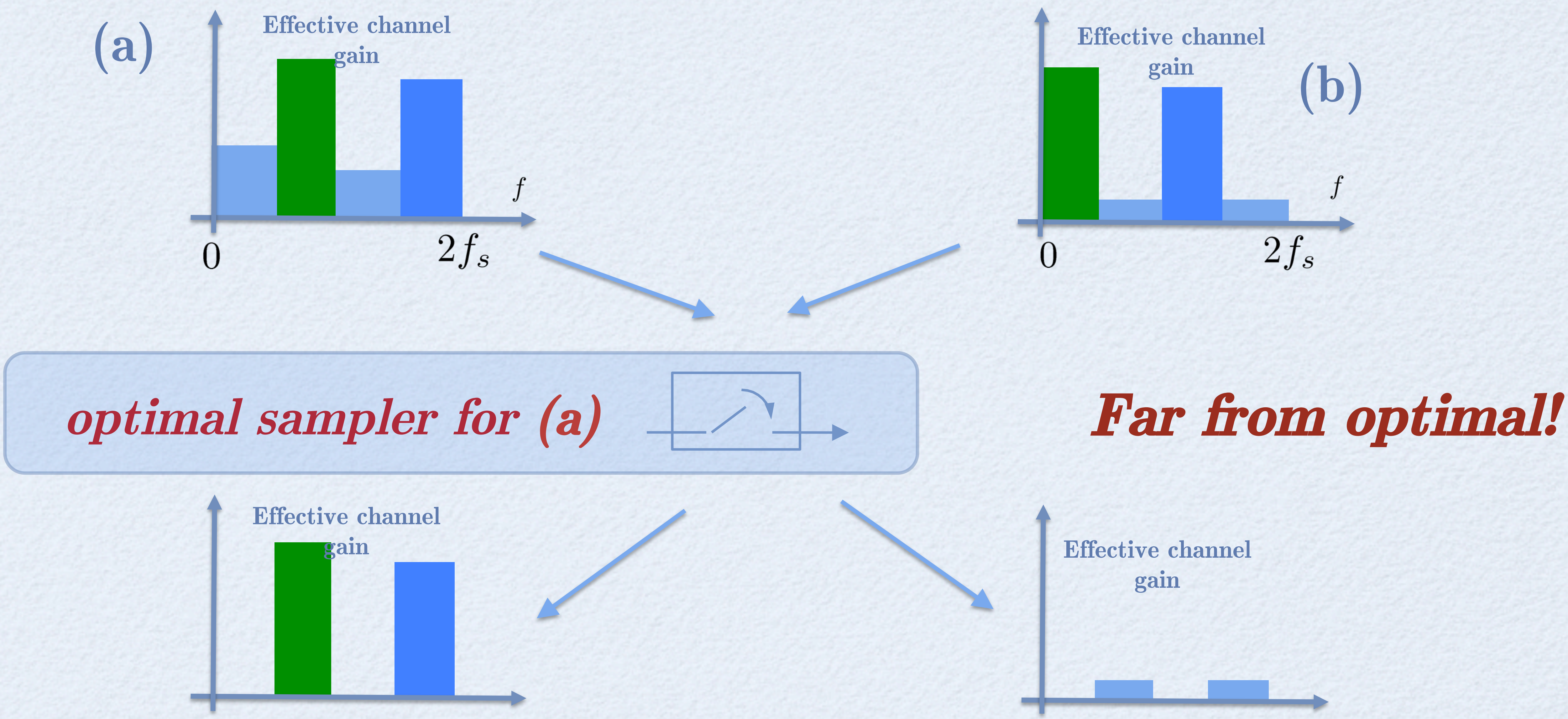

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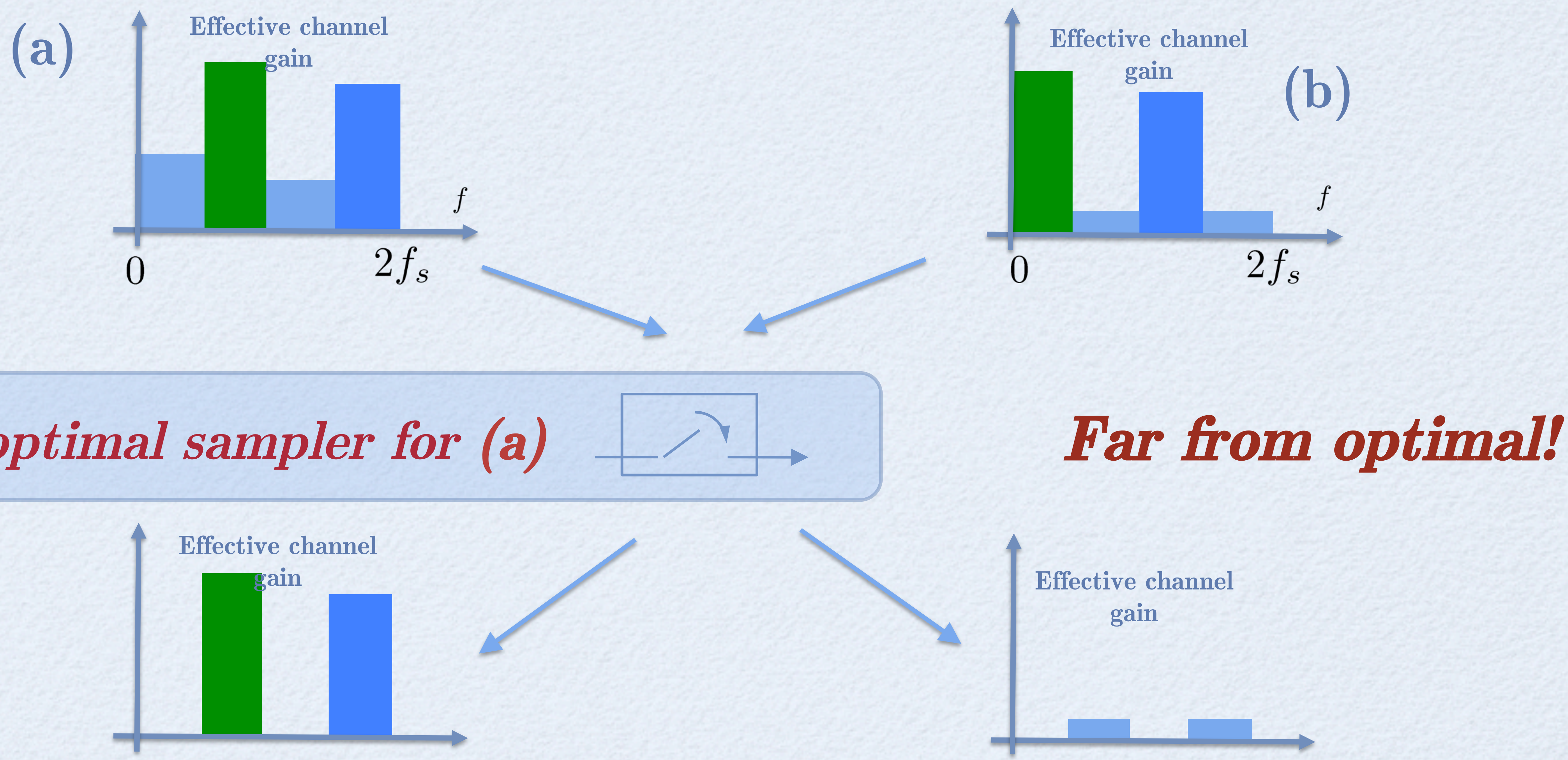
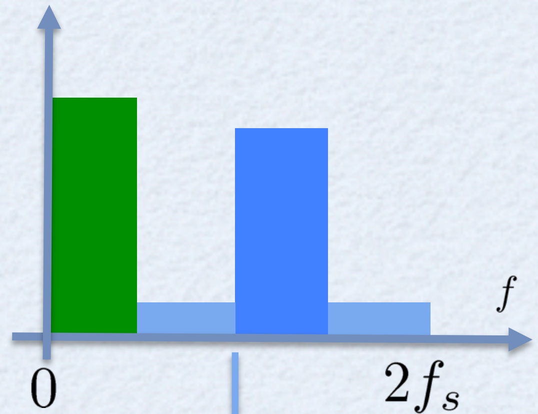
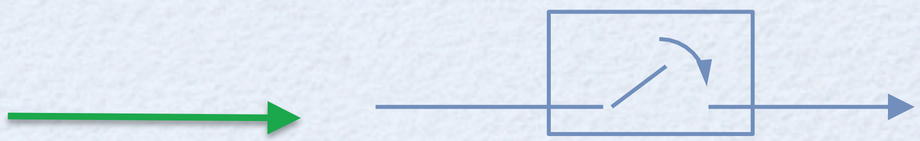


- **No single linear sampler can maximize capacity for all realizations!**
- **Question:** how to design a universal sampler ***robust to different channel realizations***

Robustness Measure: Minimax Capacity Loss

- Consider a channel state *s* and a sampler *Q* :





maximum capacity C_s

achievable rate under Q: C_s^Q

Capacity Loss: $L_s^Q := C_s - C_s^Q$

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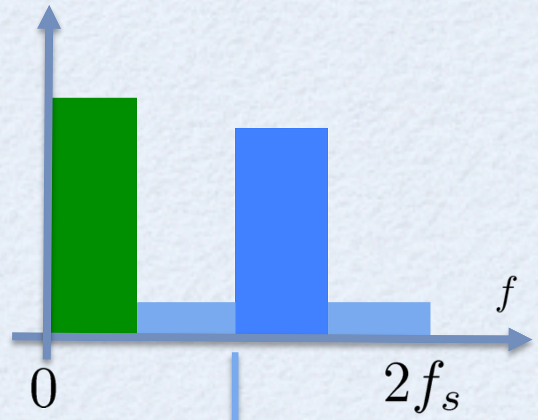

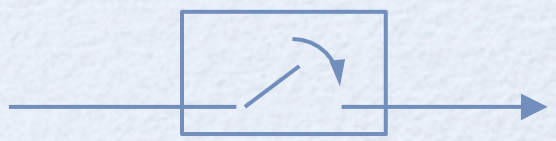

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accounting for all channel states *s*

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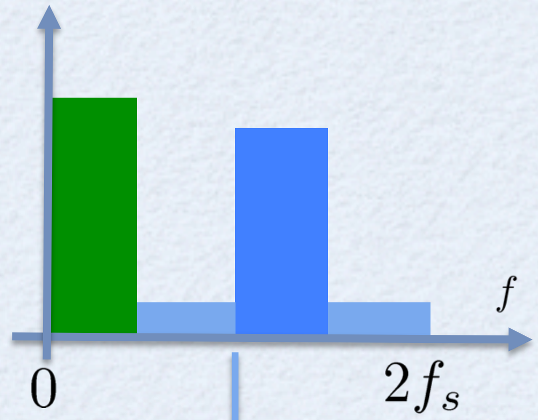

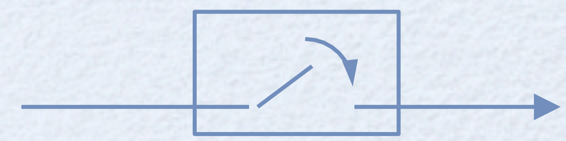

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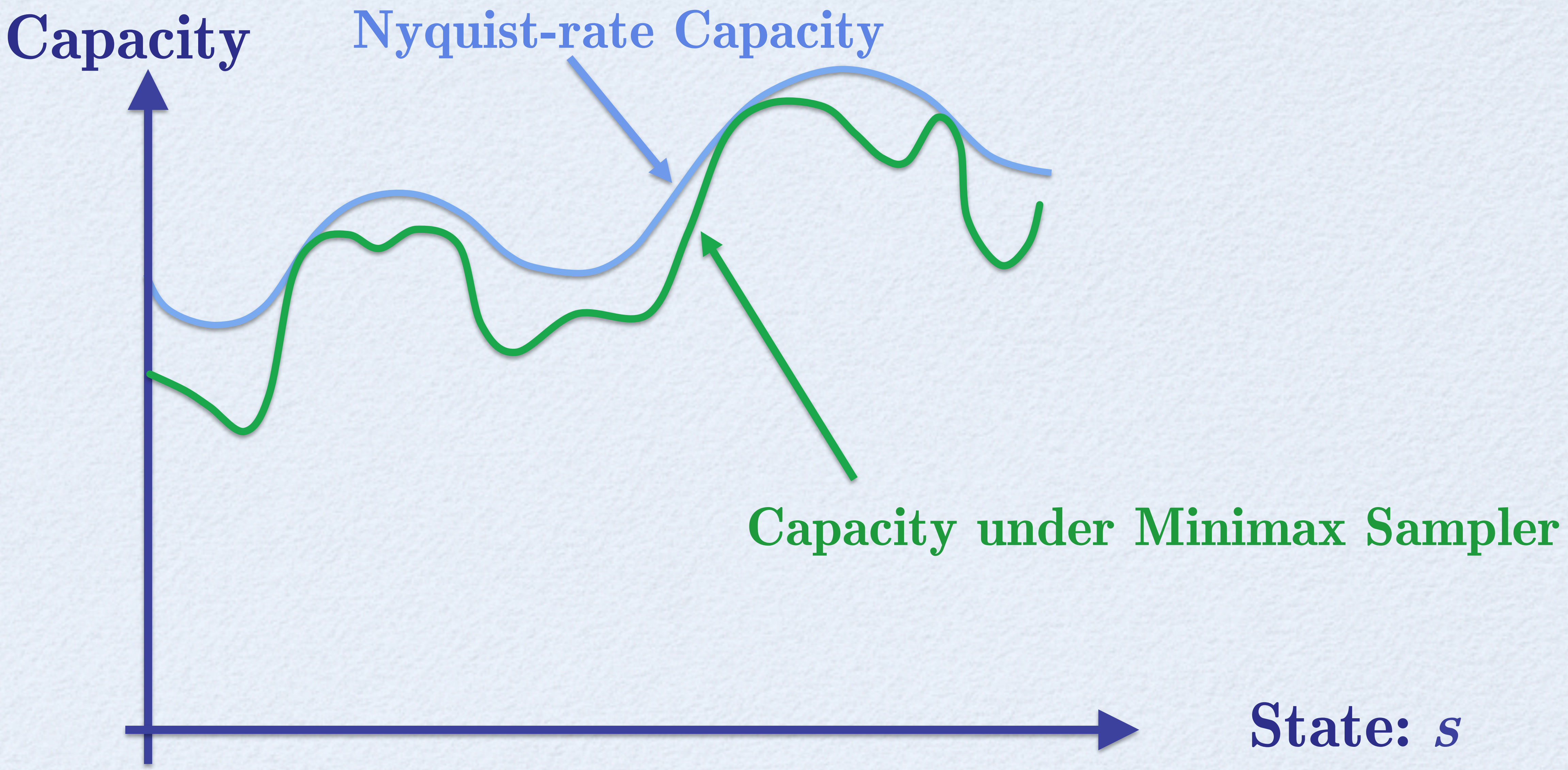
optimize over a large class of samplers

accounting for all channel states *s*

$$Q^* = \arg\min_Q \max_s L_s^Q$$ -- **Minimax Sampler**

Minimax Universal Sampling



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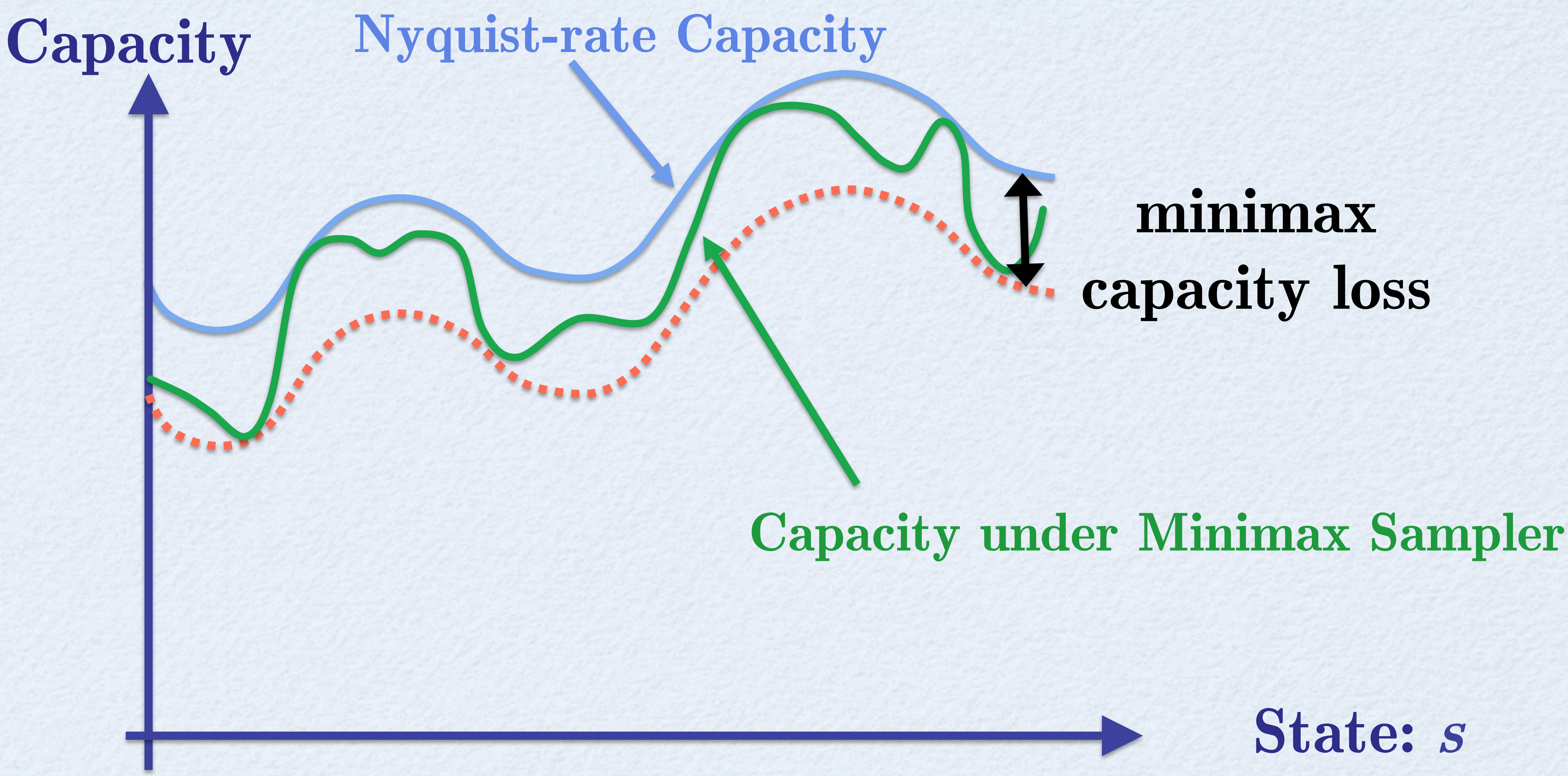

– A sampler that minimizes the worse-case capacity loss due to universal sampling

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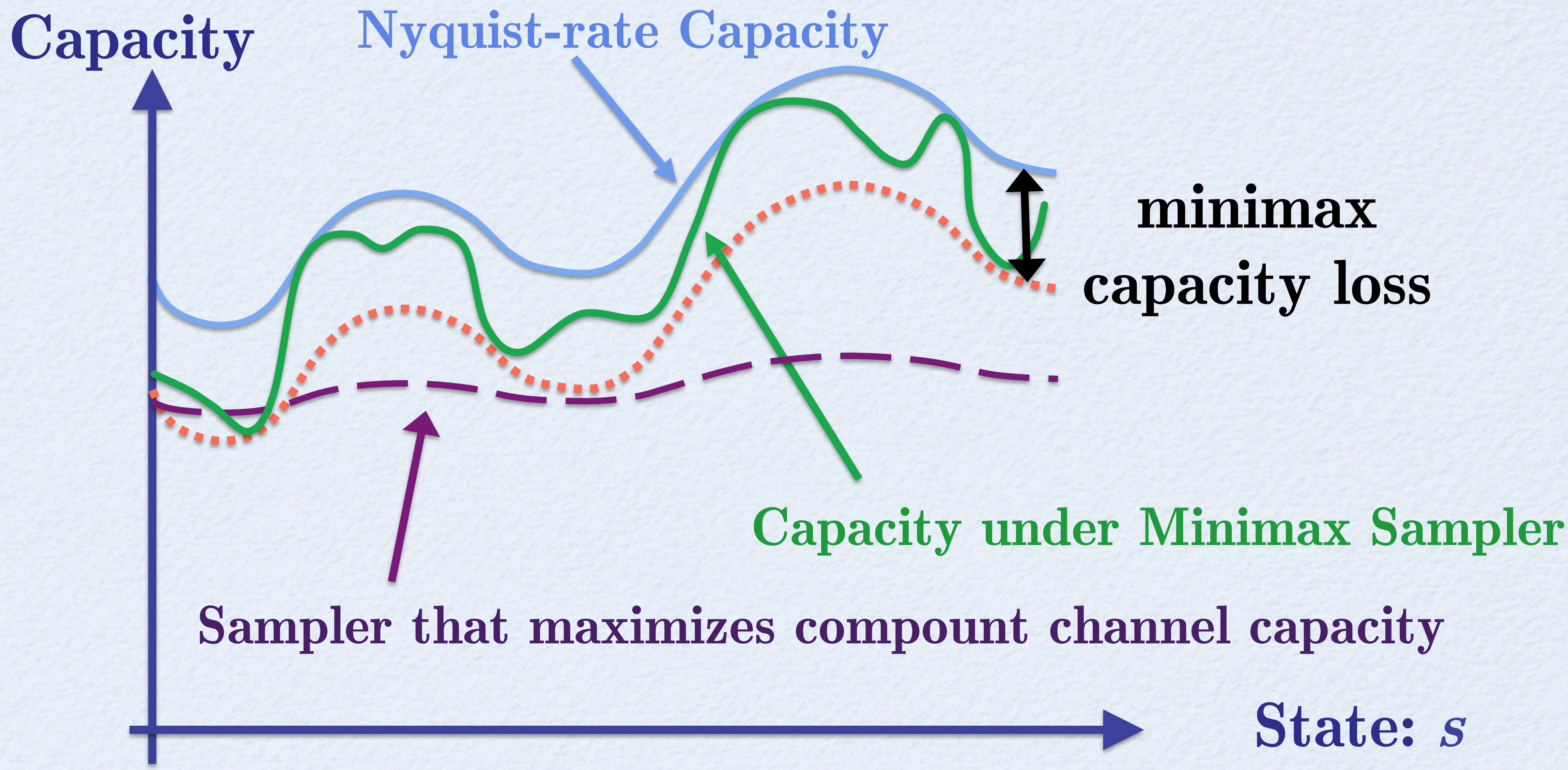
Minimax Universal Sampling

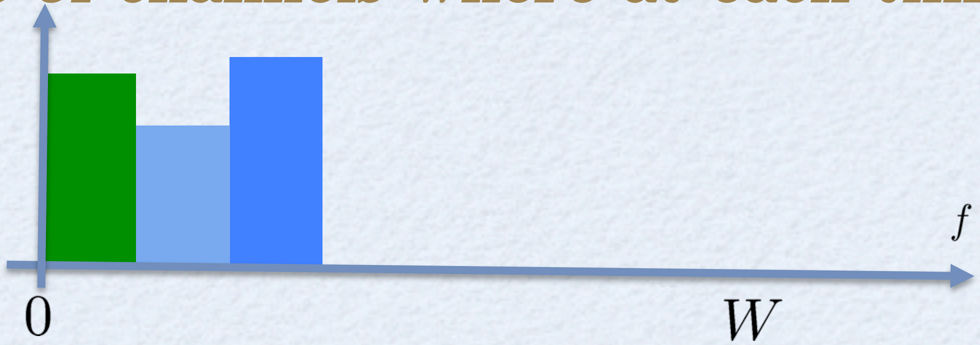


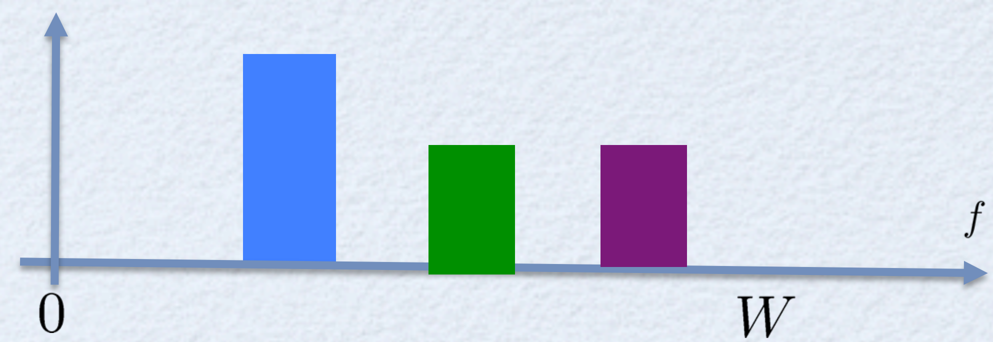
-- A sampler that maximizes compound channel capacity $\hat{Q} = \arg\max_Q \min_s C_s^Q$

– A sampler that minimizes the worse-case capacity loss due to universal sampling $Q^* = \arg\min_Q \max_s C_s - C_s^Q$

Focus on Multiband Channel Model

A class of channels where at each time only a fraction of bandwidths are active.





k **out of** ***n*** **subbands are active.**

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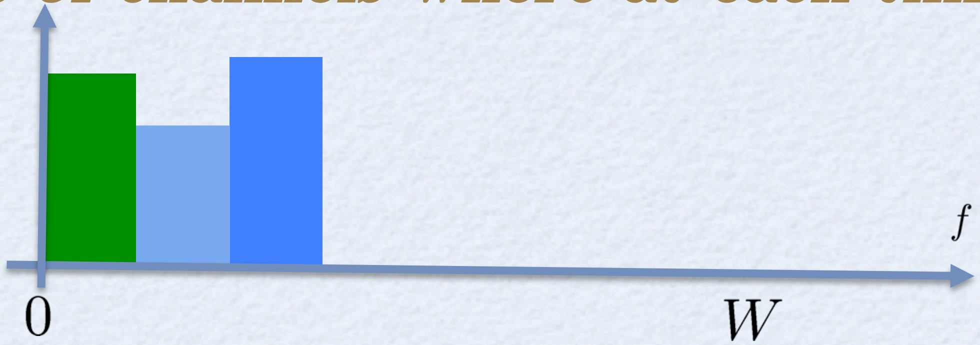

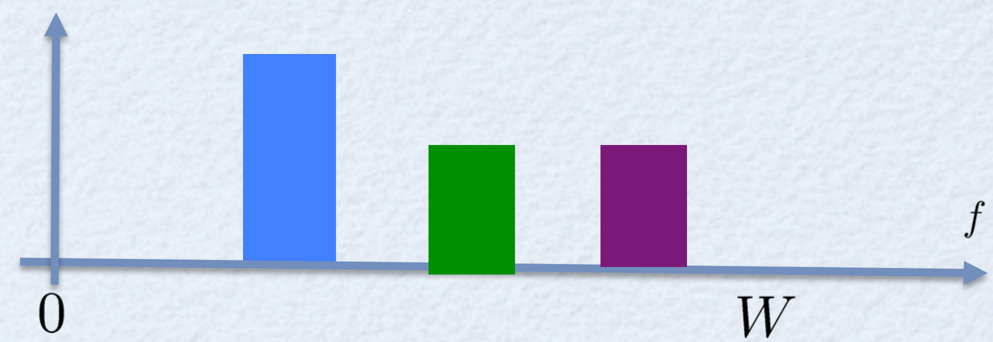

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Sparsity ratio: $\beta := k/n$

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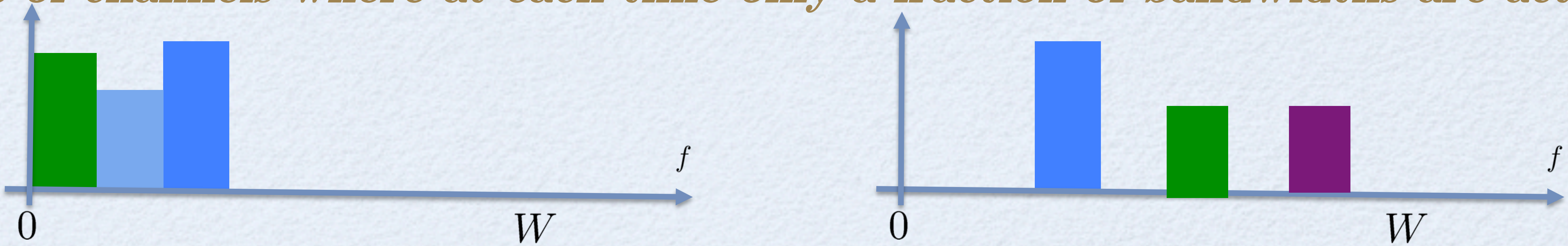

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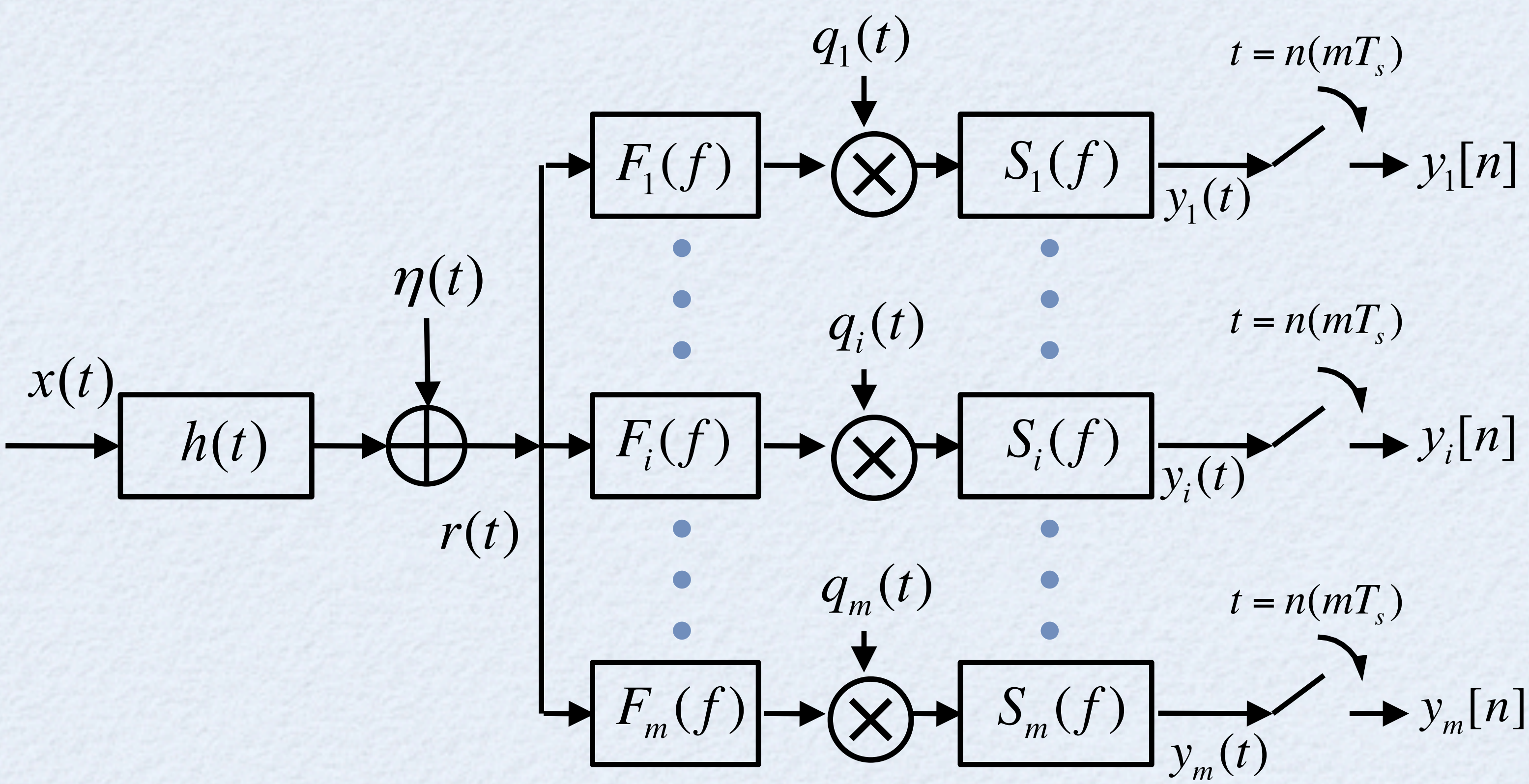
m-branch sampling with modulation and filtering:



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Converse: Landau-rate Sampling ($\alpha=\beta$)

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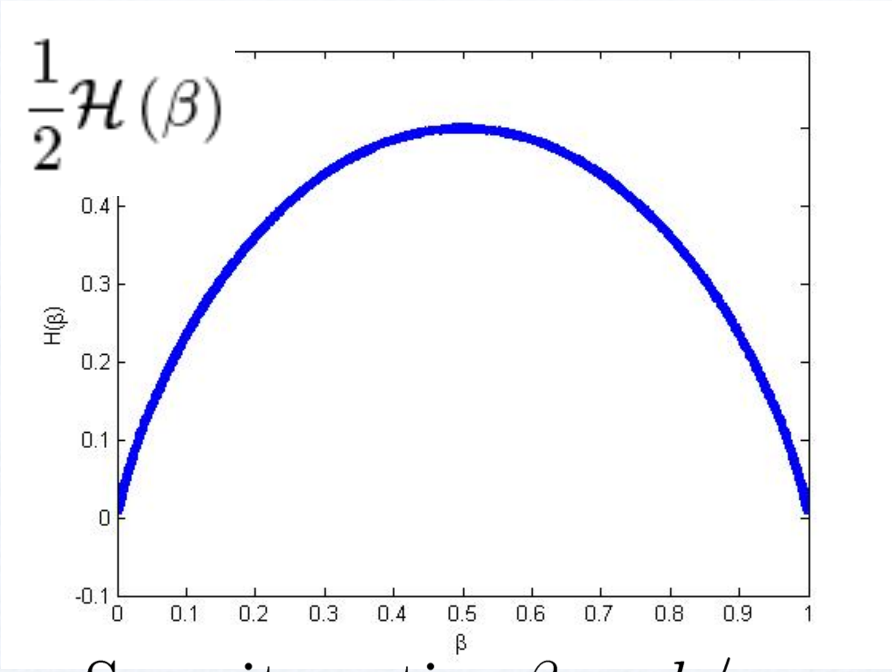
Theorem (***Converse***): The minimax capacity loss ***per Hertz*** obeys:

$$\inf_{Q} \max_{s\in\binom{[n]}{k}} L_s^Q \geq \frac{1}{2}\left\{\mathcal{H}(\beta) - \frac{2}{\sqrt{\mathrm{SNR}_{\min}}} - \frac{\log n}{n}\right\}$$

$$\mathcal{H}(\beta) := -\beta\log\beta - (1-\beta)\log(1-\beta)$$



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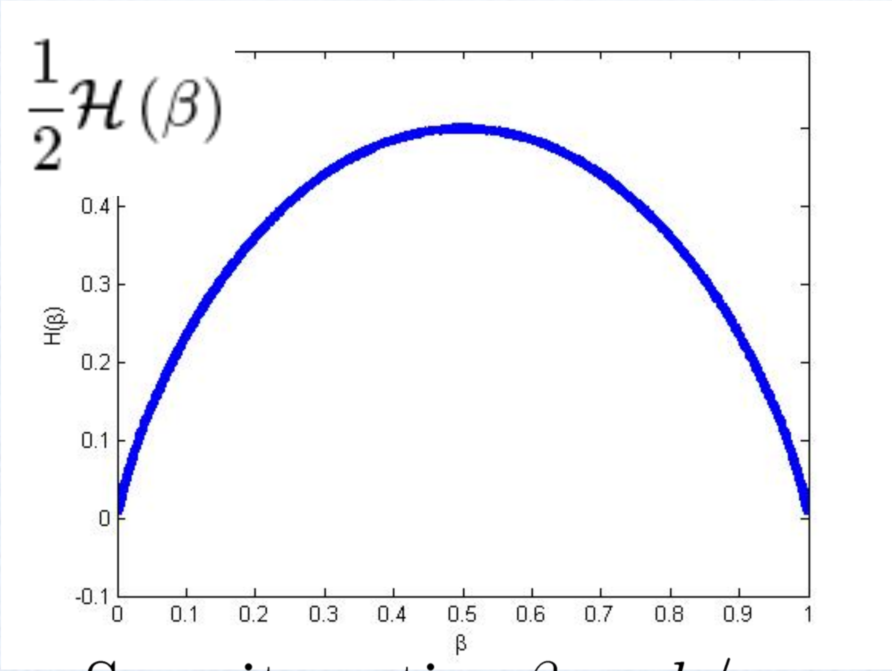

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At high SNR and large n,

minimax capacity loss determined by subband uncertainty

Converse: Landau-rate Sampling (α=β)

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Key observation for the proof :

$$\sum_s \exp(L_s^Q) \approx \text{constant}$$

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*The minimax sampler achieves **equivalent loss** across all channel states*

Achievability: Landau-rate Sampling (α=β)

$$\text{Sparsity ratio: } \beta := k/n$$

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- ***Deterministic** optimization is NP-hard (non-convex).*

Achievability: Landau-rate Sampling ($\alpha=\beta$)

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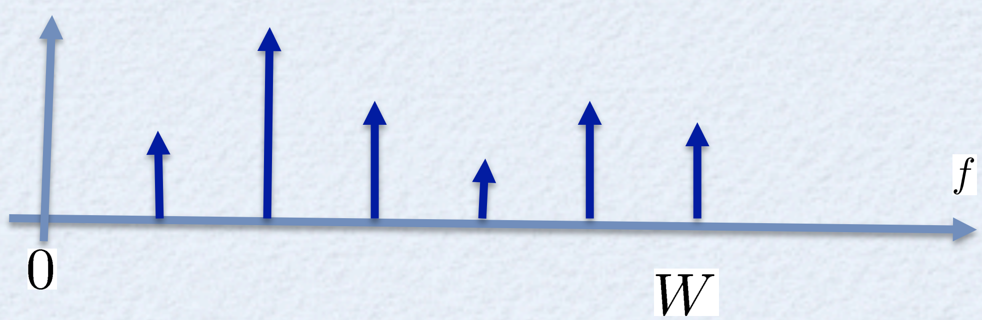
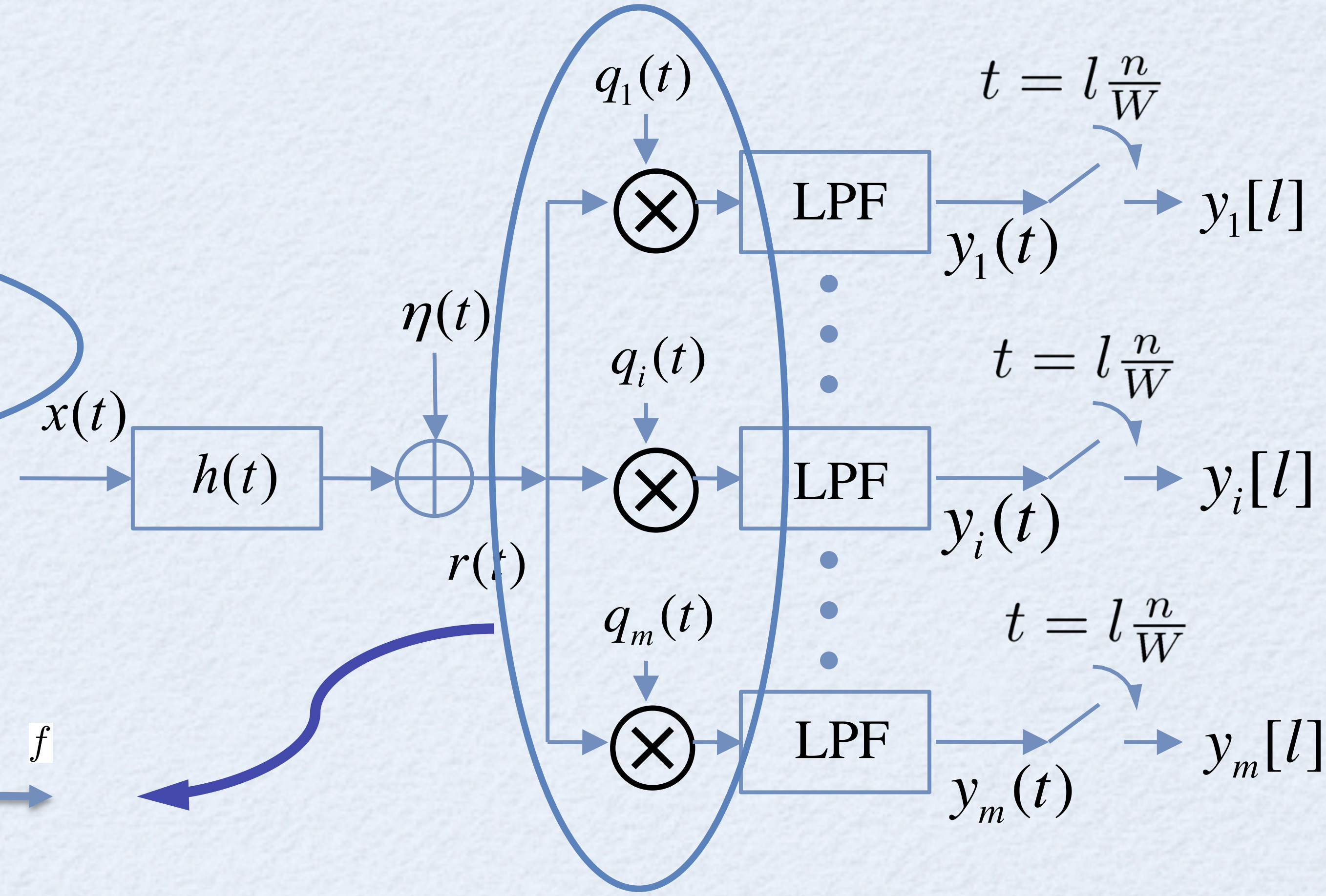
- ***Deterministic** optimization is NP-hard (non-convex).*
- **Hope: random sampling**

Fourier transform of periodic sequence is a spike-train





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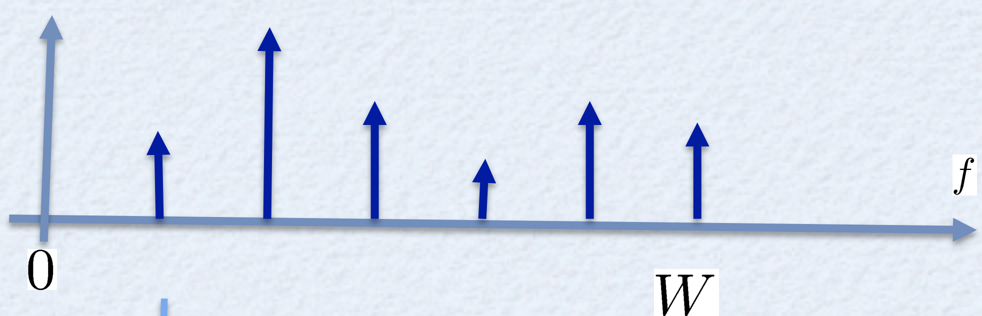

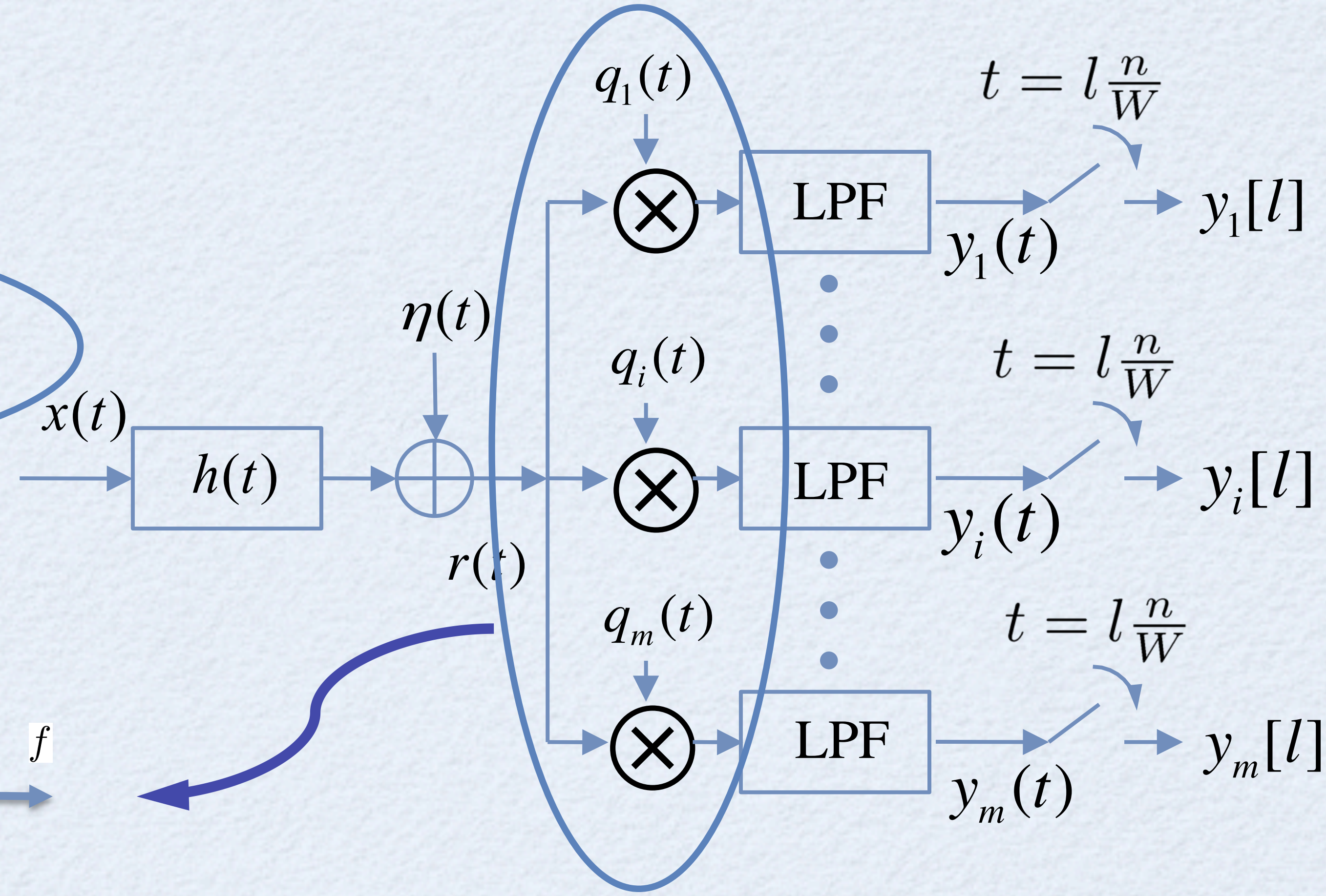

A sampling system is called ***independent random sampling*** **if** ***the coefficients of the spike-train are independently and randomly generated.***

Achievability: Landau-rate Sampling (α=β)

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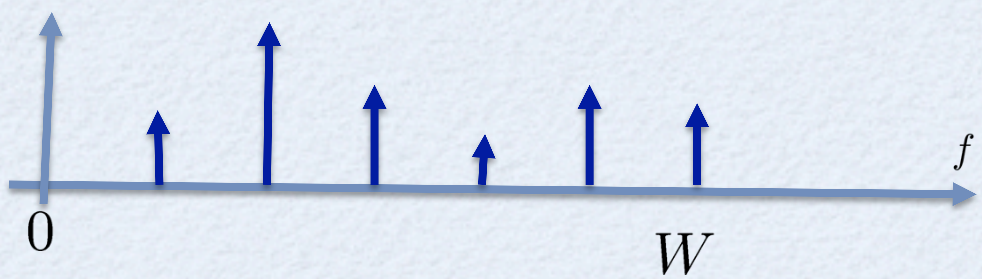
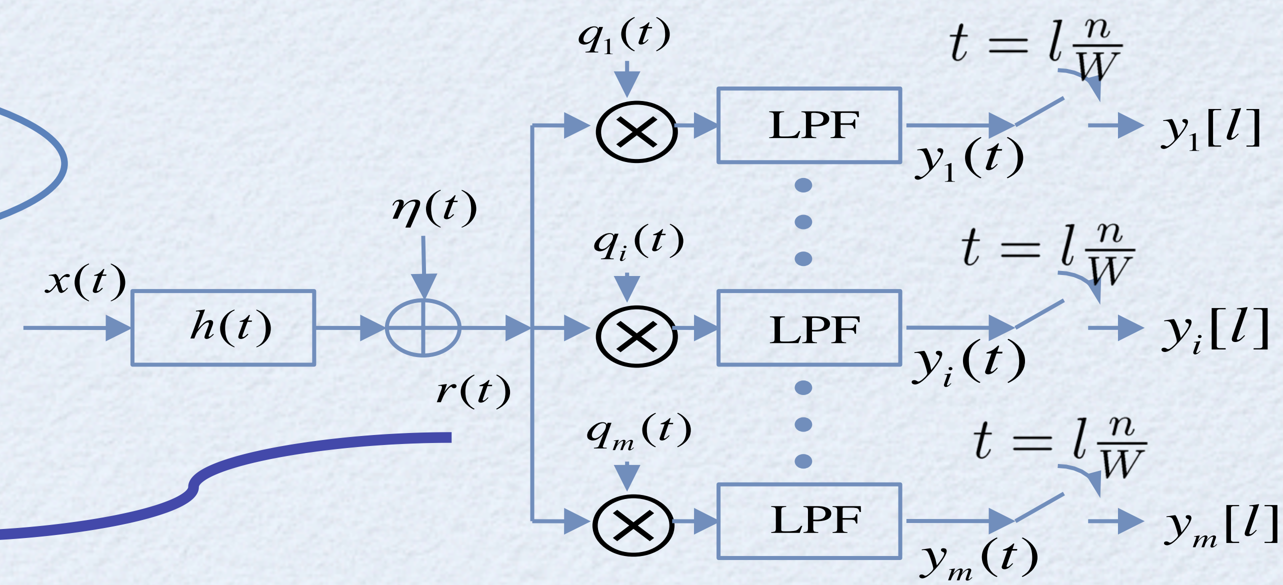
random sampling
→ ***random modulation coefficients***





Theorem (*Achievability*): The capacity loss *per Hertz* under independent random sampling is

$$\forall s \in \binom{[n]}{k}: \quad L_s^Q \leq \frac{1}{2}\left\{\mathcal{H}(\beta) + \frac{5\log k}{n} + \frac{\beta}{\mathrm{SNR}_{\min}}\right\}$$

with probability exceeding $1 - e^{-\Omega(n)}$.

Implications: Landau-rate Sampling (α=β)

Theorem (*Converse*):

$$\inf_{Q} \max_{s \in \binom{[n]}{k}} L_s^Q \geq \frac{1}{2}\left\{\mathcal{H}(\beta) - \frac{2}{\sqrt{\mathrm{SNR}_{\min}}} - \frac{\log n}{n}\right\}$$

Theorem (*Achievability*): Under **independent random sampling** (with zero mean and unit variance), with ***exponentially*** high probability,

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- Random sampling is ***Minimax***
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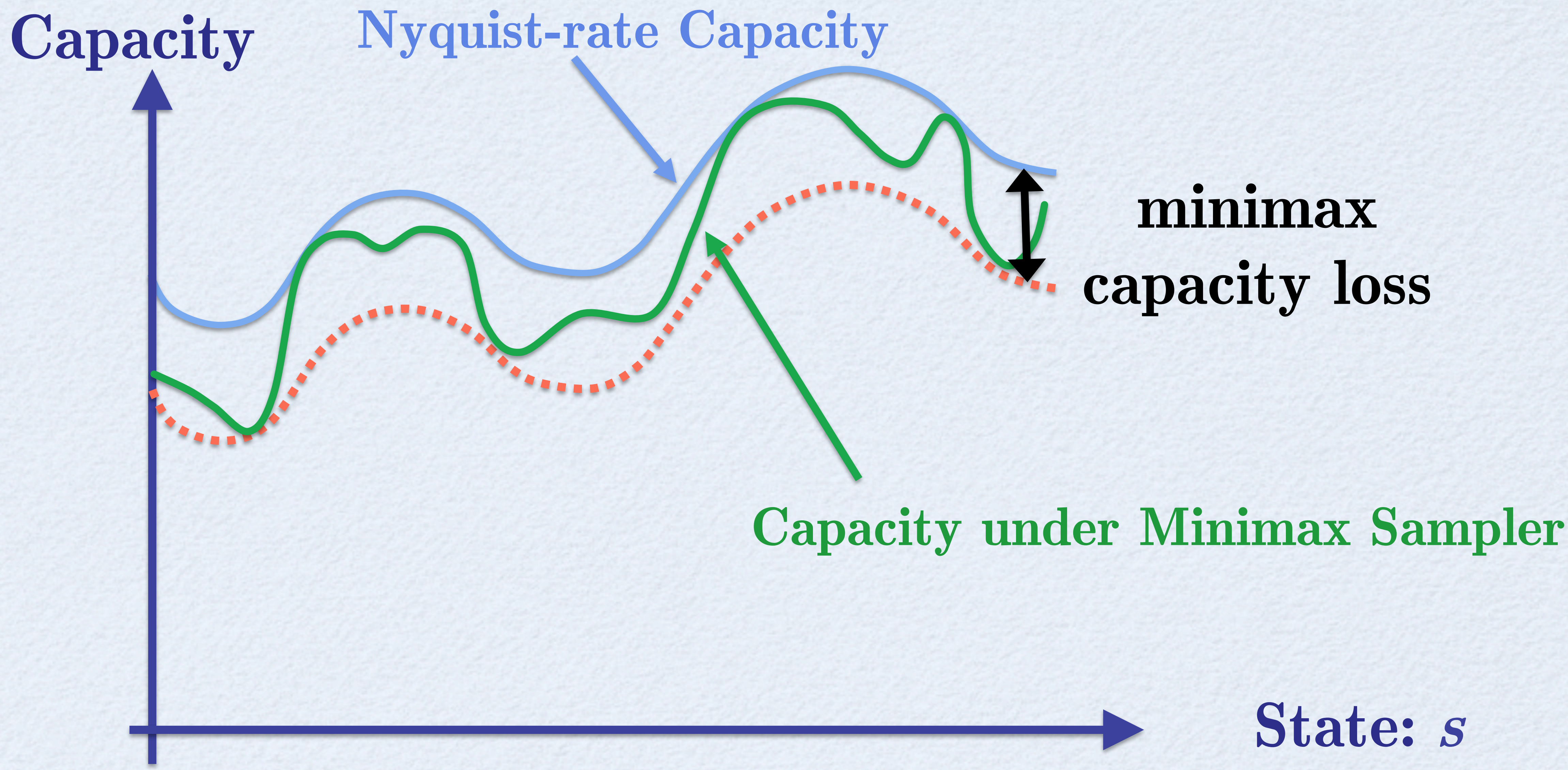
- Random sampling is ***Minimax***
- Sharp concentration – exponentially high probability
- **Universality phenomena**:
  - *A large class of distributions can work!*

    -- Gaussian, Bernoulli, uniform…
  - *No need for i.i.d. randomness*

    -- can be a mixture of Gaussian, Bernoulli, uniform…

Capacity Loss for Multiband Channels



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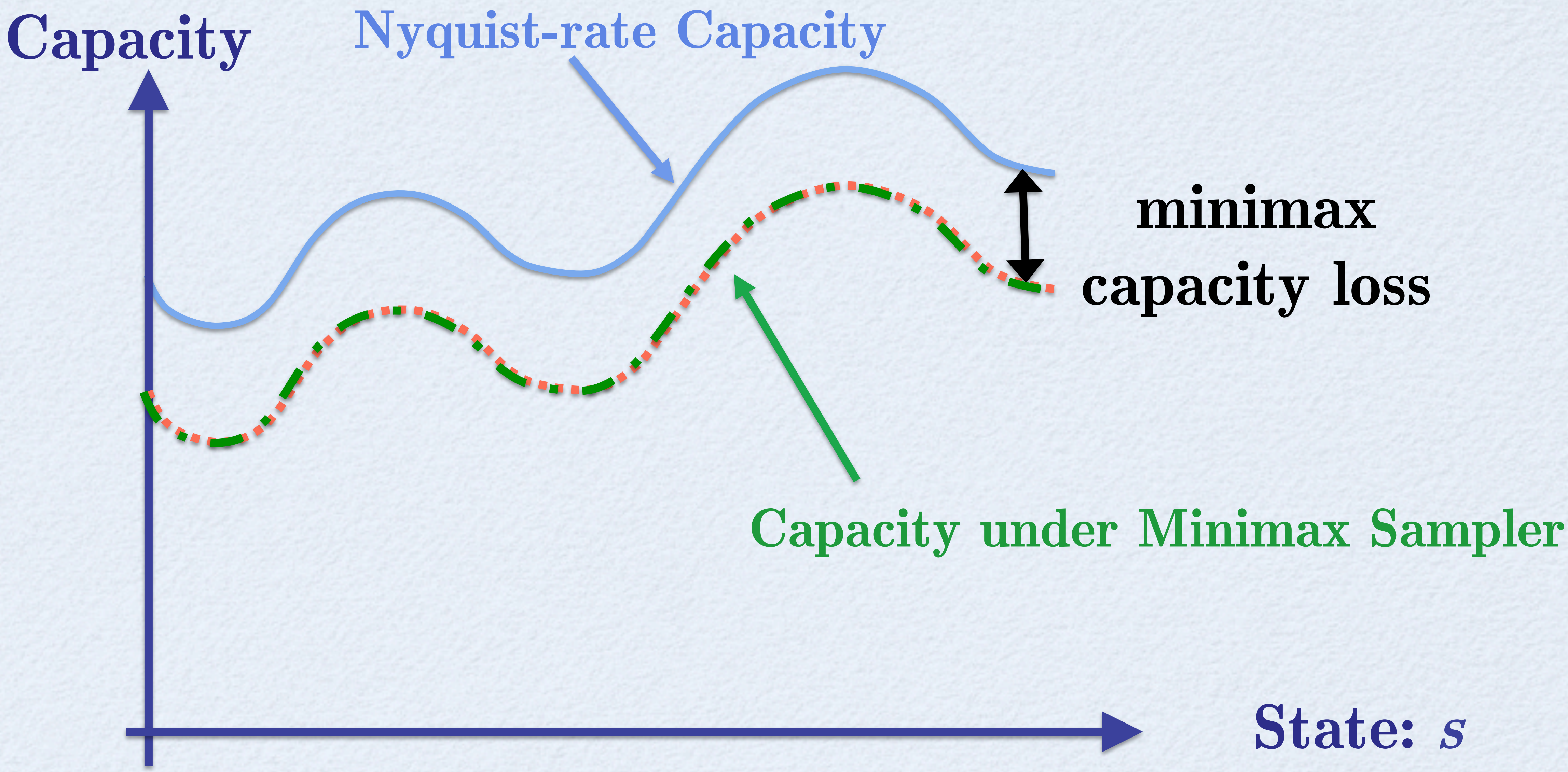


Minimax sampling yields *equivalent capacity loss* over all possible channel realizations when SNR and n are large!

Converse: Super-Landau Sampling ($\alpha > \beta$)

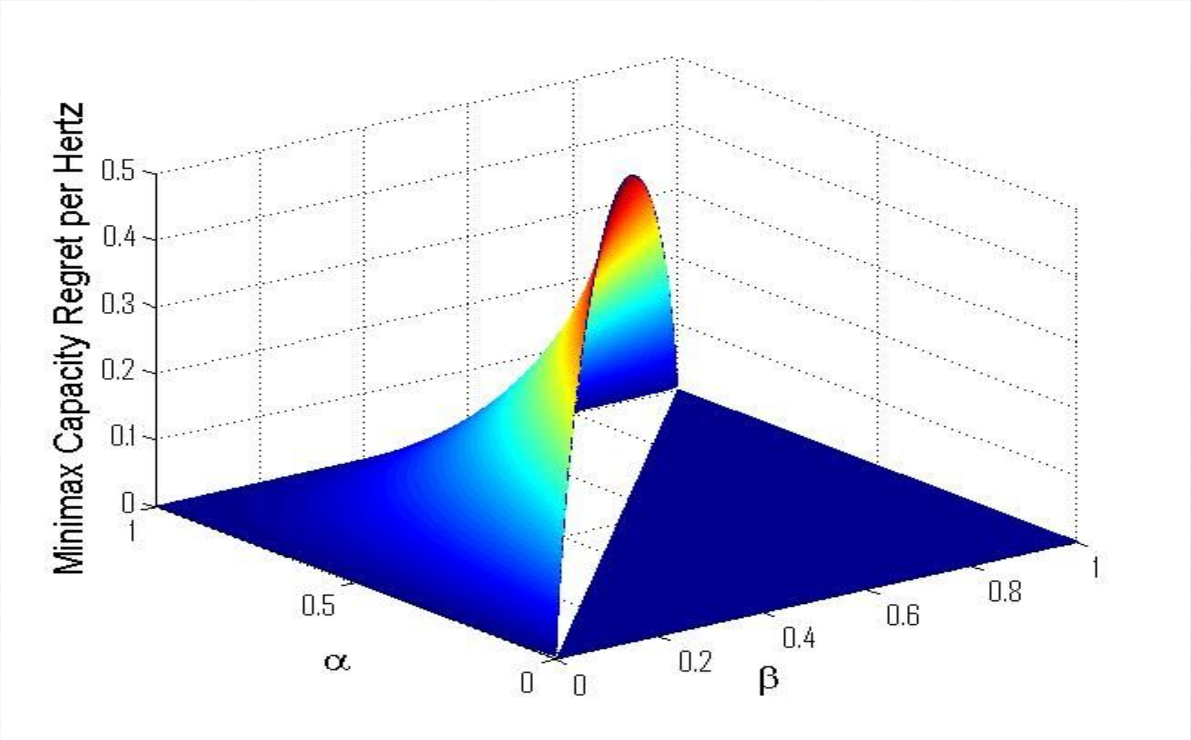
Sparsity ratio: $\beta := k/n$

Undersampling ratio: $\alpha := m/n = f_s/W$

Theorem (*Converse*): The minimax capacity loss ***per Hertz*** obeys:

$$\inf_{Q} \max_{s \in \binom{[n]}{k}} L_s^Q \geq \frac{1}{2}\left\{ \mathcal{H}(\beta) - \alpha \mathcal{H}\left(\frac{\beta}{\alpha}\right) - \frac{2}{\sqrt{\mathrm{SNR}_{\min}}} - \frac{\log n}{n} \right\}$$

- Capacity gain due to oversampling is $\frac{1}{2}\alpha\mathcal{H}\left(\frac{\beta}{\alpha}\right)$



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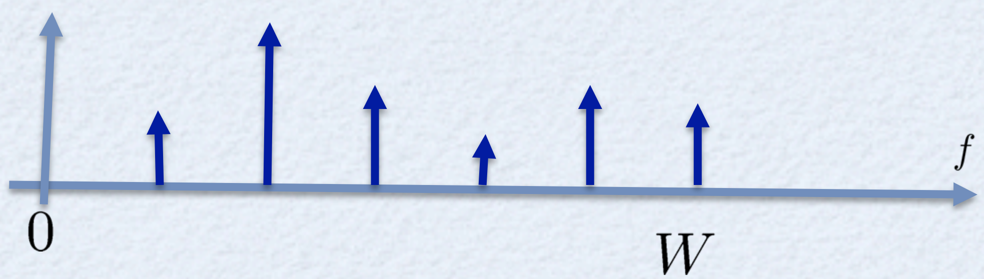
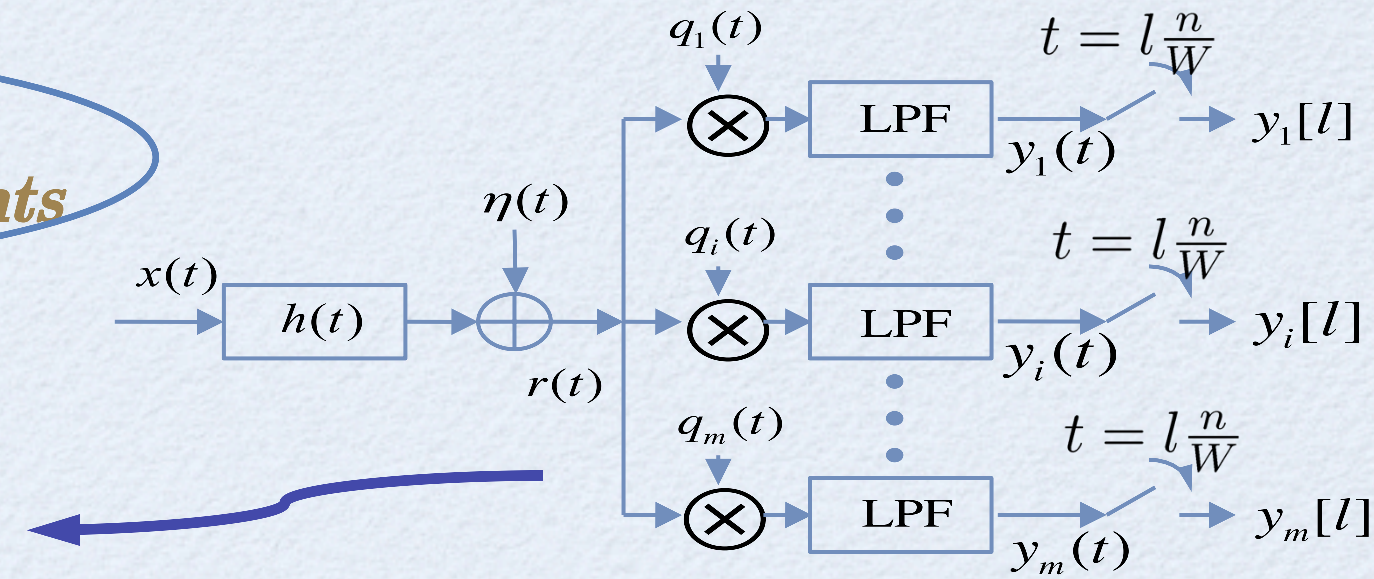
Gaussian sampling

→ *Gaussian modulation coefficients*





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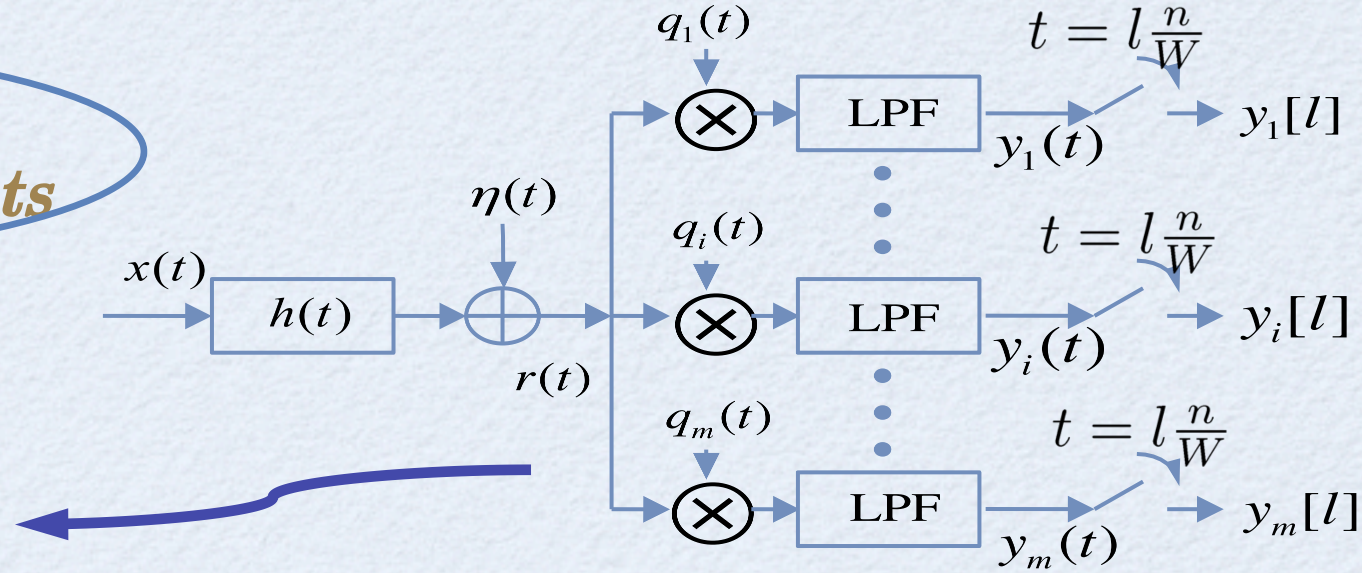
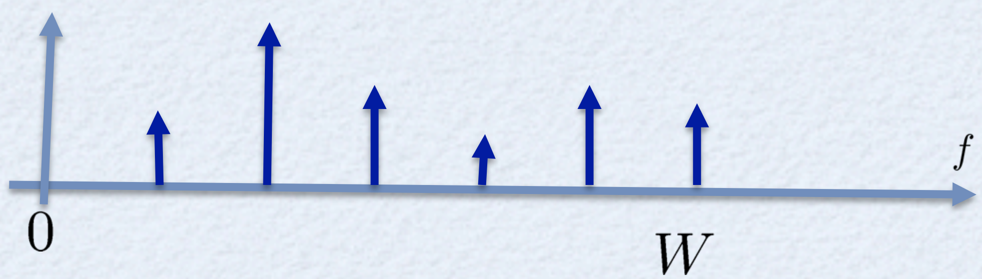
Gaussian sampling
→ Gaussian modulation coefficients





Theorem (***Achievability***): If $\alpha+\beta<1$, then the capacity loss ***per Hertz*** under **i.i.d. Gaussian random sampling** is

$$\forall s \in \binom{[n]}{k}: \quad L_s^Q \le \frac{1}{2}\left\{ \mathcal{H}(\beta) - \alpha \mathcal{H}\left(\frac{\beta}{\alpha}\right) + O\left(\frac{\log^2 n}{\sqrt{n}}\right) + \frac{\beta}{\mathrm{SNR}_{\min}} \right\}$$

with probability exceeding $1 - e^{-\Omega(n)}$.

Implications: super-Landau sampling (α=β, α+β<1)

Theorem (*Converse*): The minimax capacity loss *per Hertz* obeys:

$$\inf_{Q} \max_{s\in\binom{[n]}{k}} L_s^Q \geq \frac{1}{2}\left\{\mathcal{H}(\beta) - \alpha\mathcal{H}\left(\frac{\beta}{\alpha}\right) - \frac{2}{\sqrt{\mathrm{SNR}_{\min}}} - \frac{\log n}{n}\right\}$$

Theorem (*Achievability*): Under **i.i.d. Gaussian random sampling**, with *exponentially* high probability

$$\forall s \in \binom{[n]}{k}: \quad L_s^Q \leq \frac{1}{2}\left\{\mathcal{H}(\beta) - \alpha\mathcal{H}\left(\frac{\beta}{\alpha}\right) + O\left(\frac{\log^2 n}{\sqrt{n}}\right) + \frac{\beta}{\mathrm{SNR}_{\min}}\right\}$$

- Gaussian sampling is ***Minimax !***
- Sharp concentration: *exponentially* high probability
- ***Universality phenomena not shown***...
  - *We have only shown the results for i.i.d. Gaussian sampling*

Concluding Remarks

- ***Minimax Capacity Loss***

  -- A new ***metric*** to characterize ***robustness against different channel realizations***

  -- For multiband channels, it depends only on undersampling factor and sparsity ratio

- ***The power of random sampling***

  -- Near-optimal in an overall sense (minimax)

  -- Large random samplers behave ***in deterministic ways***

  (sharp concentration + universality)

- ***A Non-Asymptotic analysis of random channels***

Full-Length Paper

- Y. Chen, A. J. Goldsmith, and Y. C. Eldar, "**Minimax Capacity Loss under Sub-Nyquist Universal Sampling**", *submitted to IEEE Trans Info Theory*, arxiv.org/abs/1304.7751, April 2013,

Thank You!