October 22, 2014

Near-Optimal Joint Object Matching via Convex Relaxation

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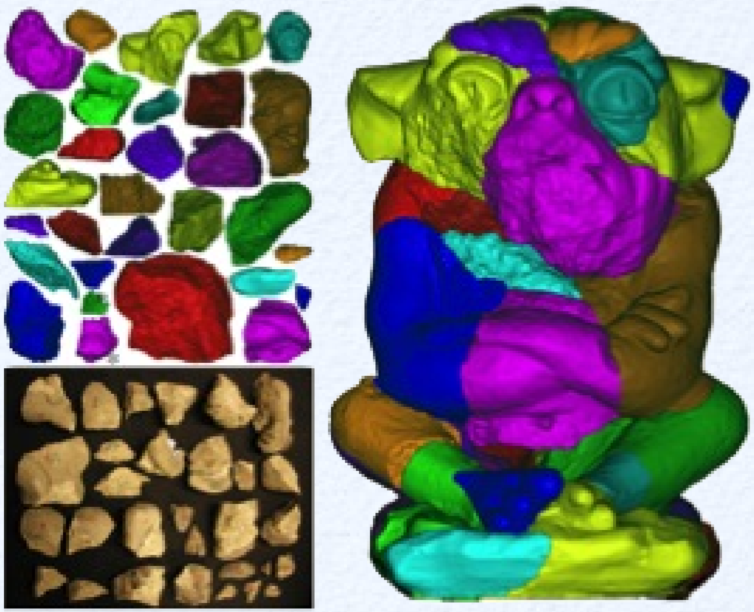
Joint Work with Qixing Huang (TTIC), Leonidas Guibas (Stanford)

Assembling Fractured Pieces

Manual Assembly (Ephesus, Turkey)

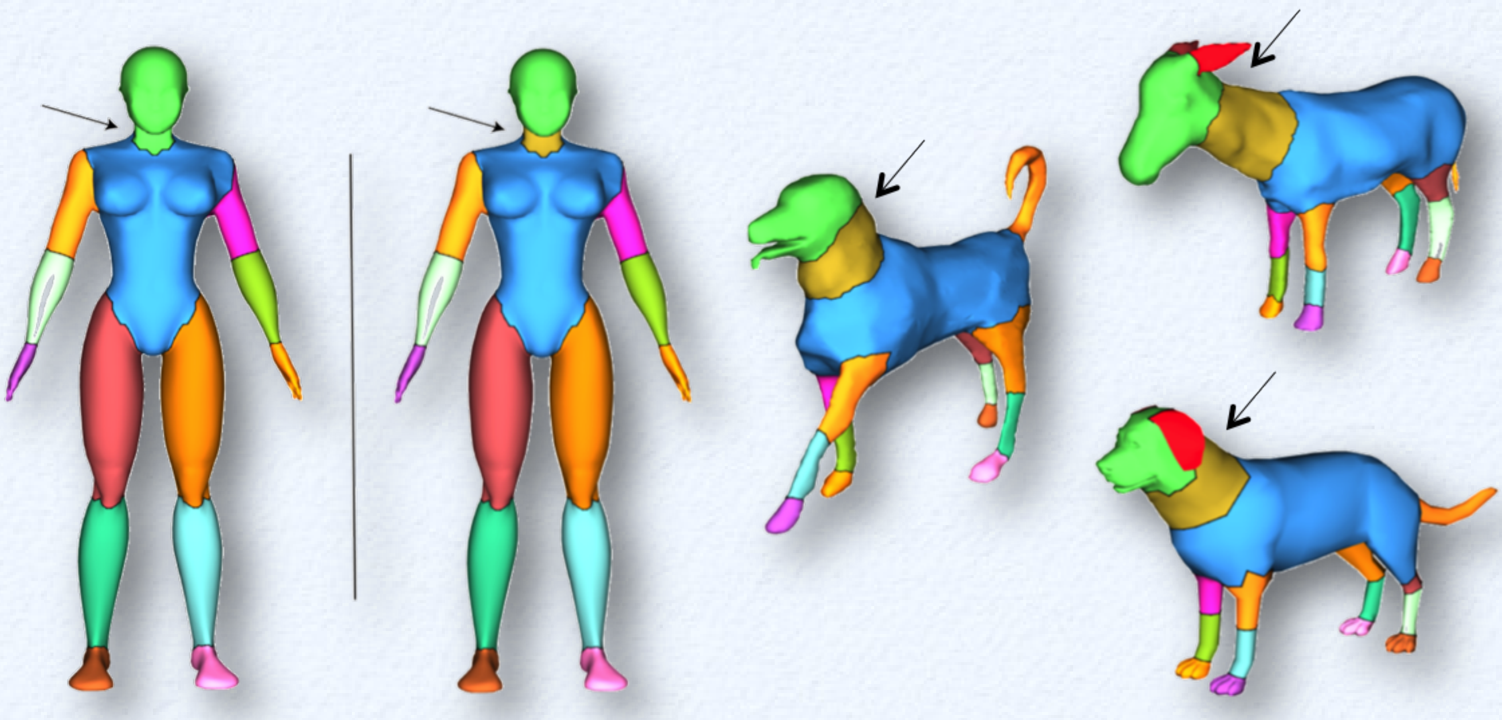


Computer Assembly
(Fig. credit: Huang et al 06)

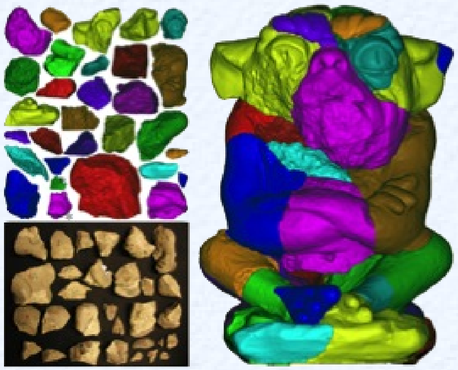
Structure from Motion from Internet Images

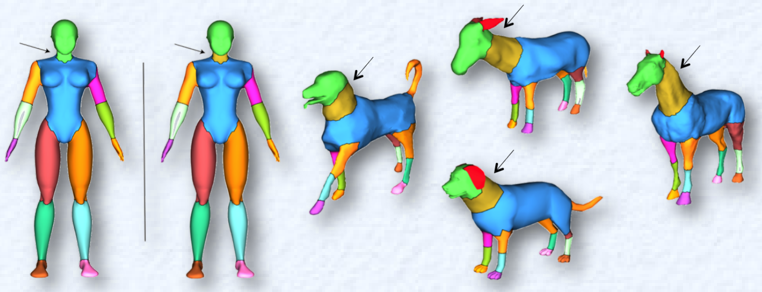
Data-Driven Shape Analysis

Example: Joint Segmentation



Joint Object/Graph Matching

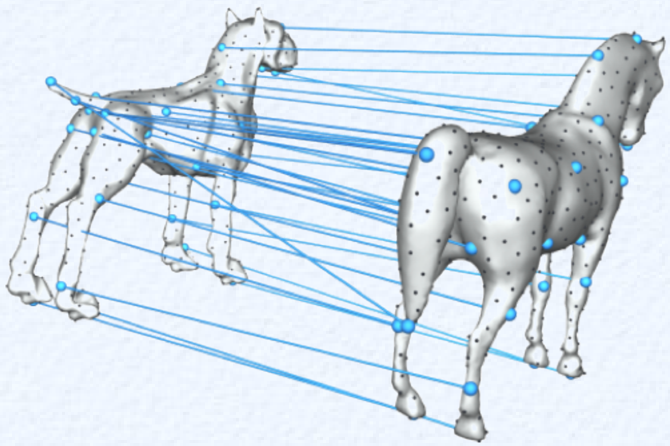




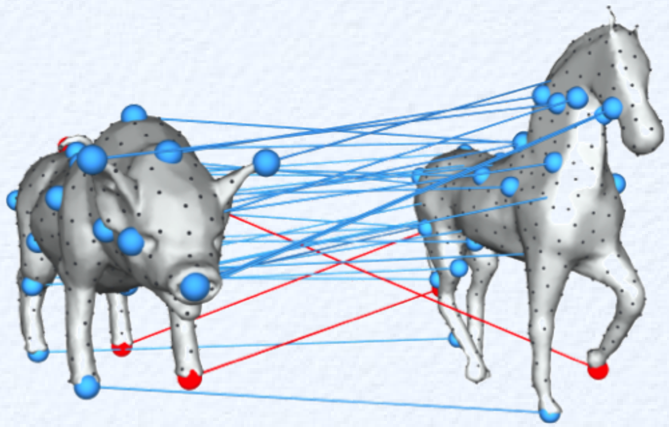
- **Given**: n objects (graphs), each containing a few elements (vertices)
- **Goal**: *consistently* match all similar elements across all objects

Naive Approach: Pairwise Matching

- **Naive Approach**
  - Compute **pairwise matching** across all pairs in isolation
  - pairwise matching: extensively explored

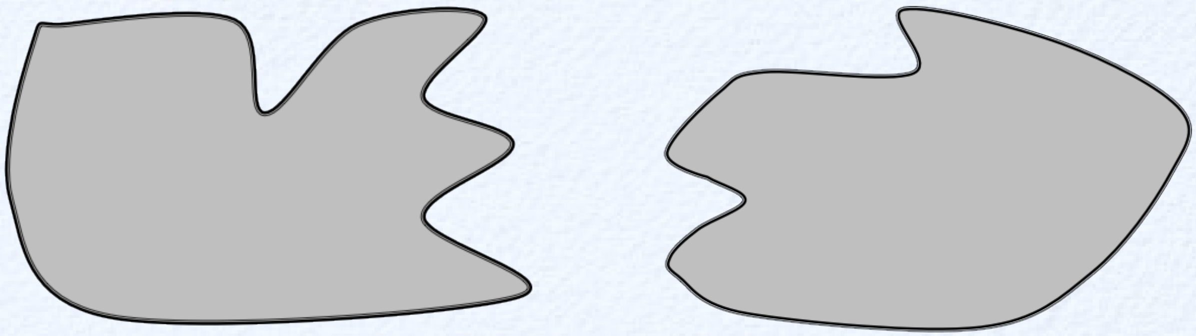


Very similar objects



Less similar objects

Are Pairwise Methods Perfect?



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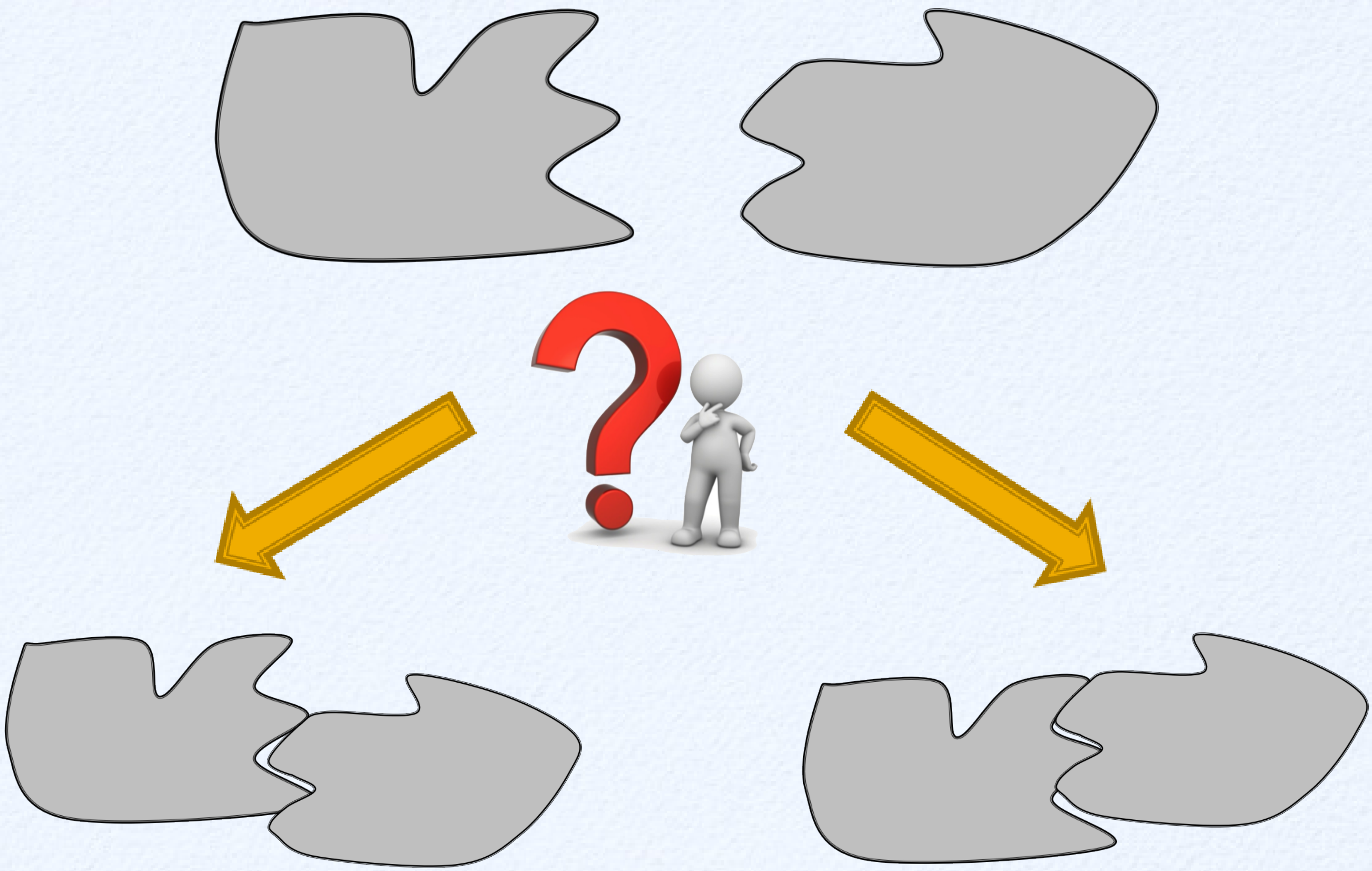
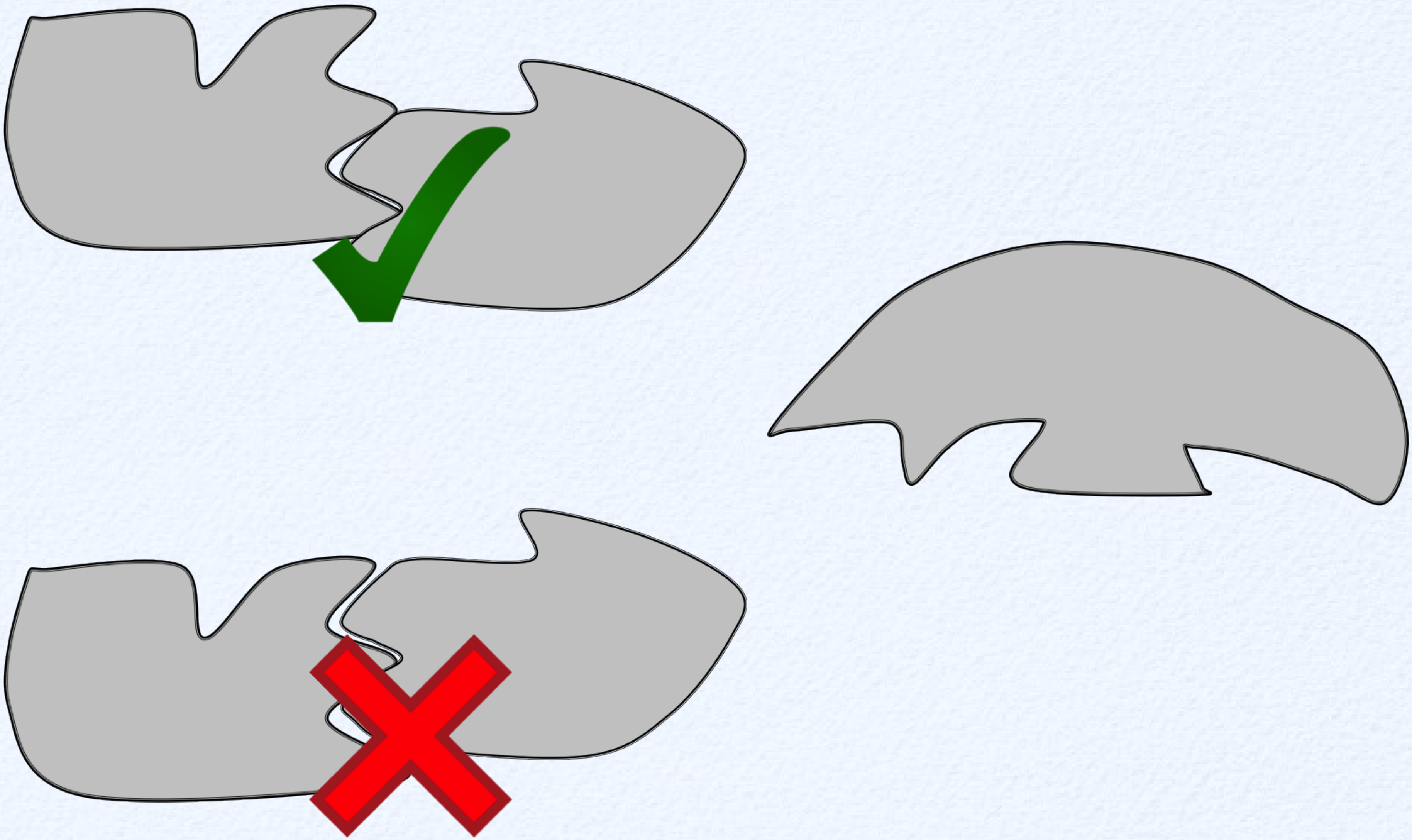


Additional Object Helps!



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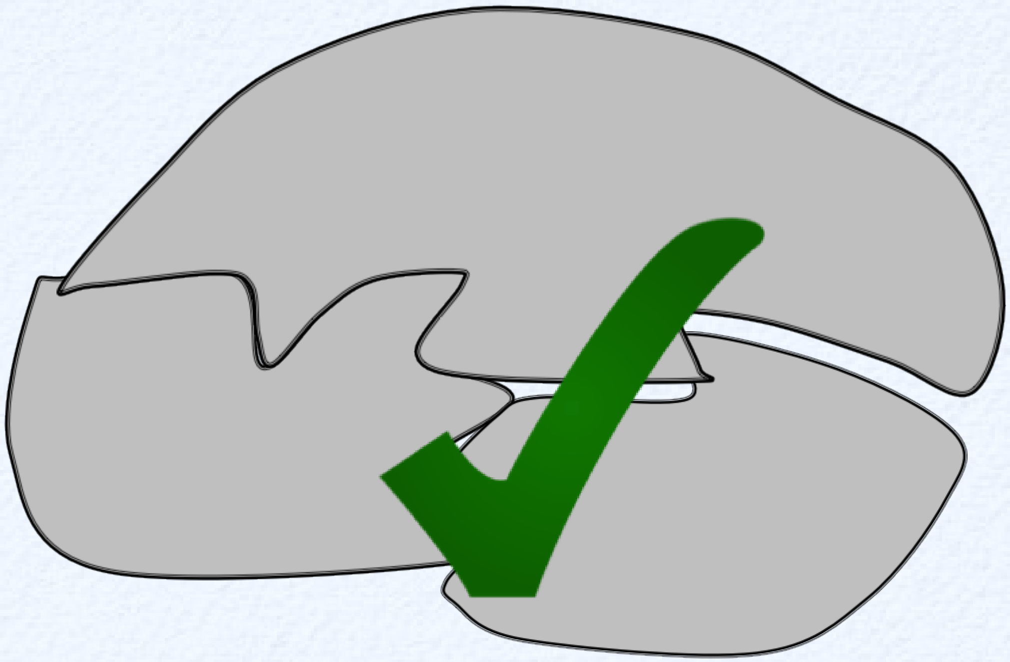

Popular Approach: 2-Stage Method

- **Stage 1: Pairwise Matching**
  - Compute pairwise matching across a few pairs **in isolation**
  - Use off-the-shelf pairwise methods

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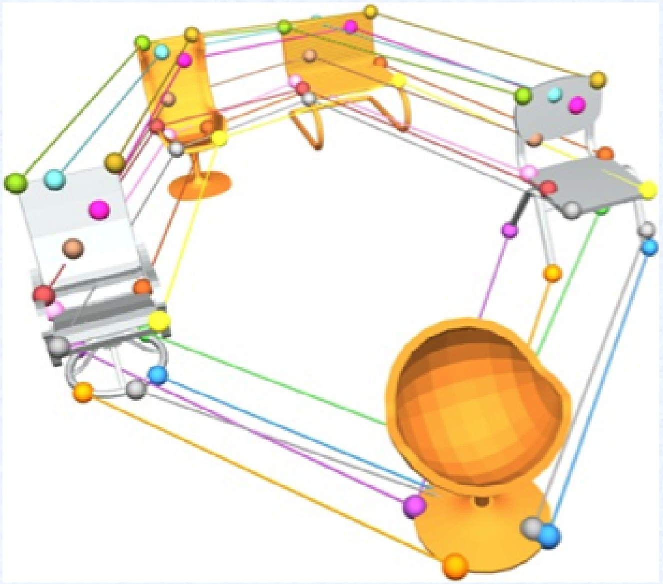
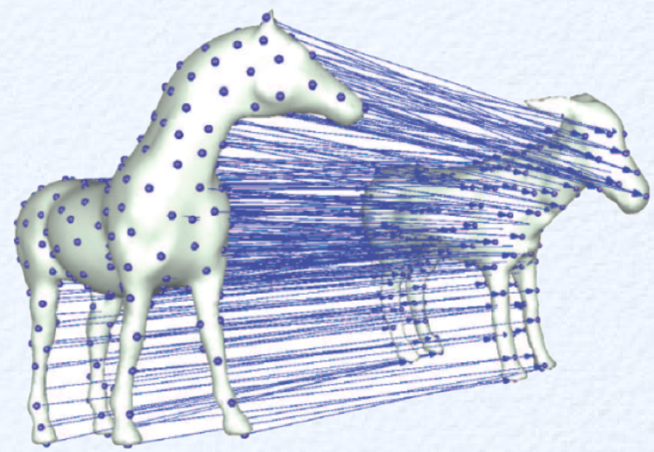


- **Stage 2: Global Refinement**
  - Jointly refine all provided maps
  - Criterion: exploit **global consistency**

Object Representation

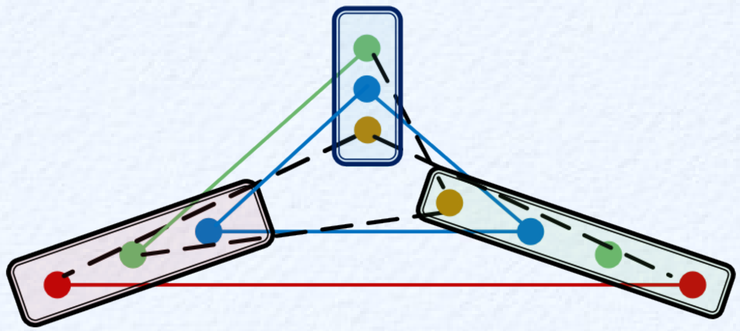
- **Object**
  - a set of points
  - drawn from the same universe

- **Map**
  - point-to-point correspondence



Problem Formulation

- **Input:** a few pairwise matches computed in isolation



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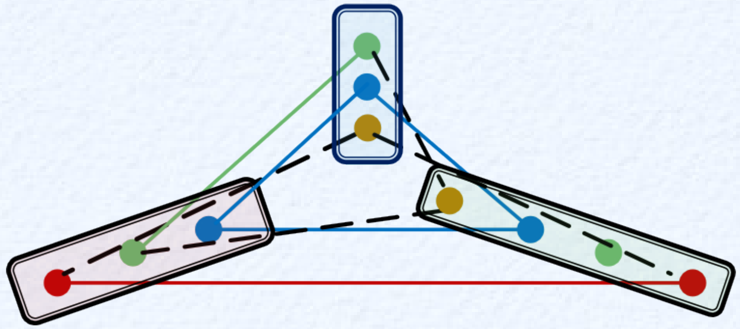
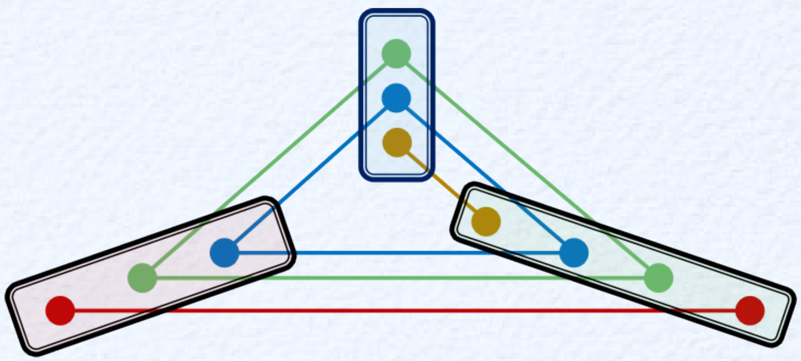
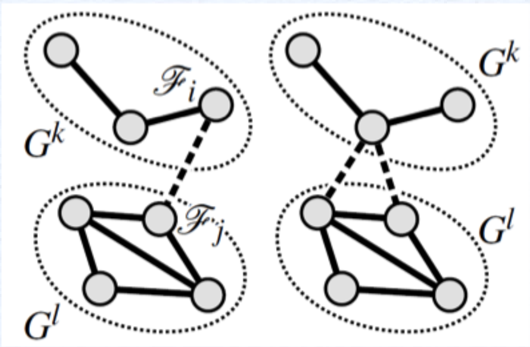


- **Output:** a collection of maps that are
  - close to the input matches
  - globally consistent

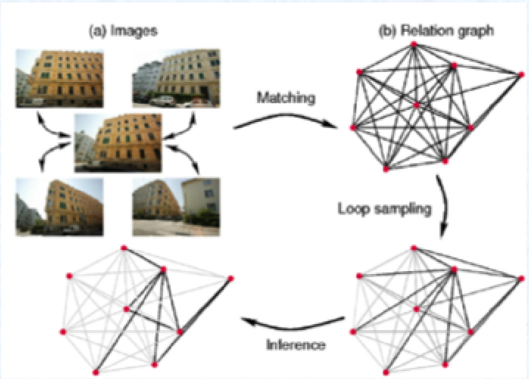


- **NP-Hard! [Huber 02]**

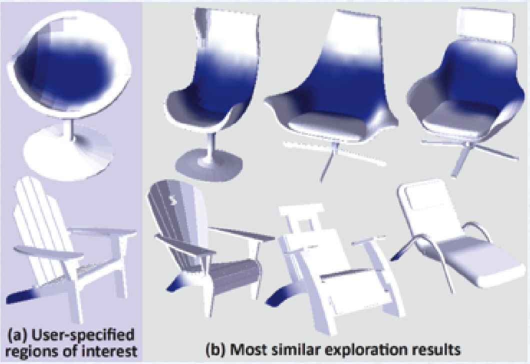
Prior Art



spanning tree optimization [Huber'02]



detecting inconsistent cycles [Zach'10, Ngu'11]



spectral technique [Kim'12, Huang'12]

- **Pros**: empirical success
- **Cons**:
  - little fundamental understanding (except [HuangGuibas'13])
  - rely on hyper-parameter tuning

Advances in Fundamental Understanding

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  - theoretical guarantees under a basic setup
  - tolerate 50% input errors

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  - recovery ability improves with # objects
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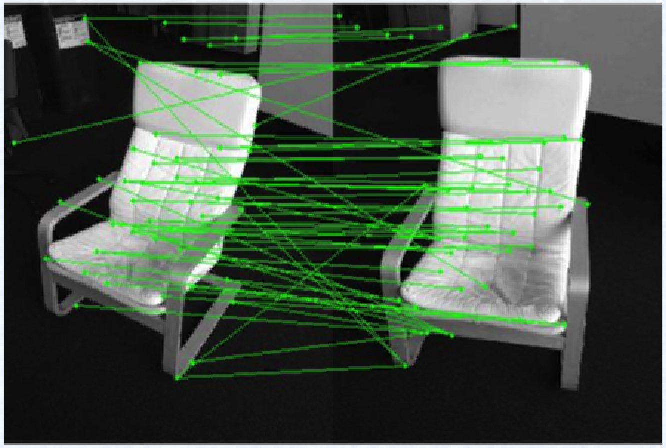
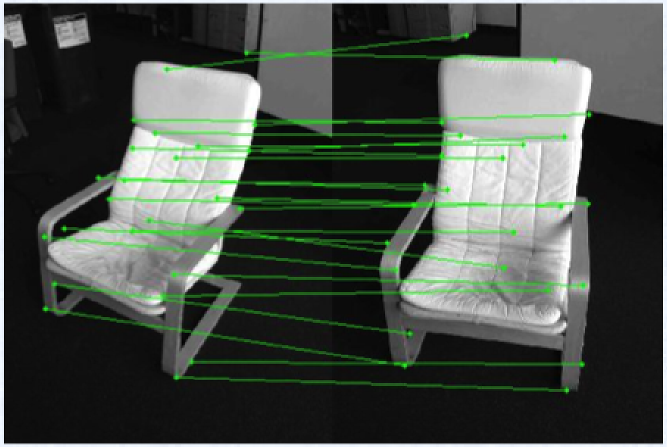
- **Several important challenges remain unaddressed...**
- **Relevant problems:**
  - rotation sync (Wang et al), multiway alignment (Bandeira et al)

Challenge 1: Dense Input Errors

- **Input Errors**
  - A significant fraction of inputs are corrupted



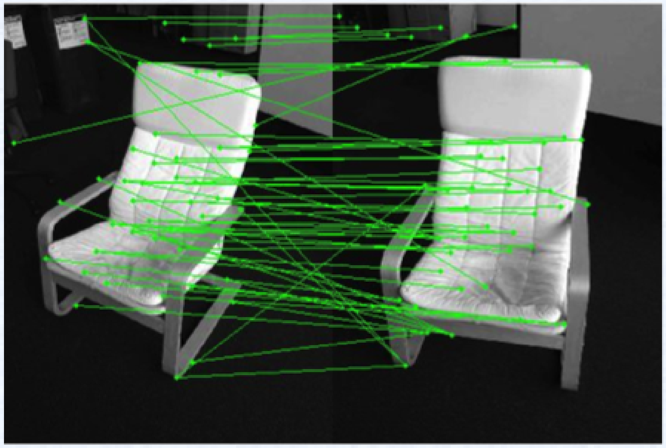
Input Maps



Ground Truth

Challenge 1: Dense Input Errors

- **Input Errors**
  - A significant fraction of inputs are corrupted
  - Prior art:
    — tolerate **50%** input errors [HuangGuibas'2013]



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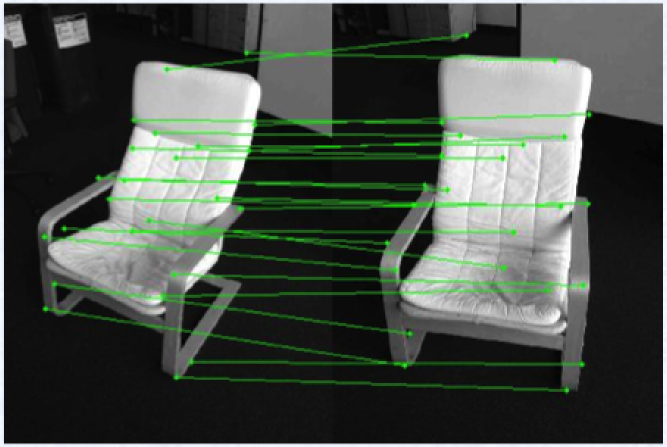
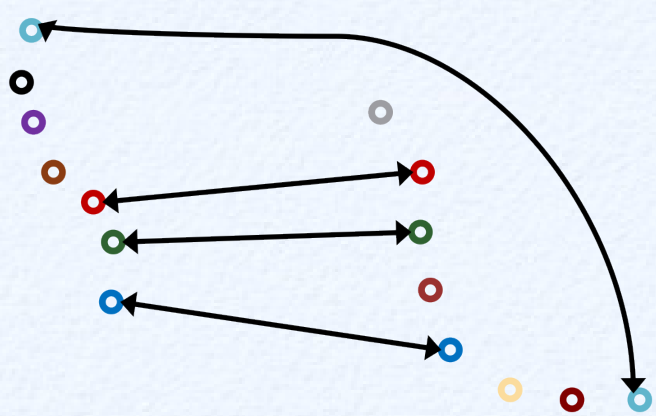


Ground Truth

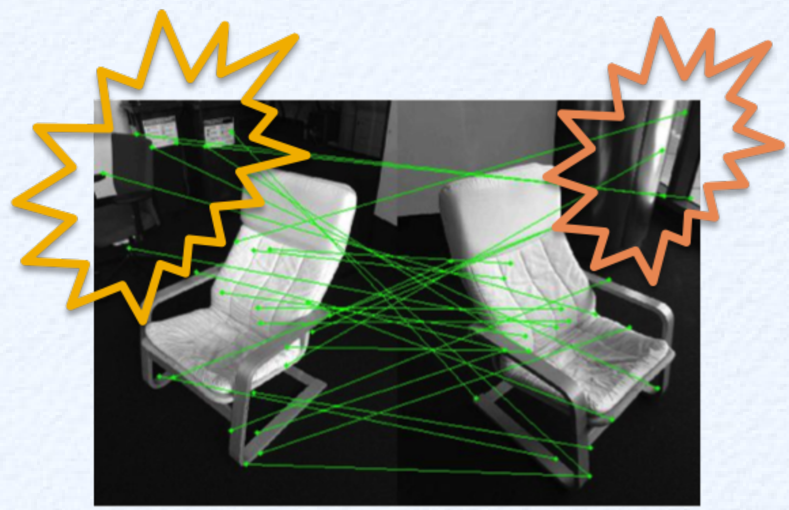
Challenge 2: Partial Similarity

- **Partial Similarity**
  - Objects might only be partially similar to each other.

    — *e.g. restricted views at different camera positions*



Subgraph Matching

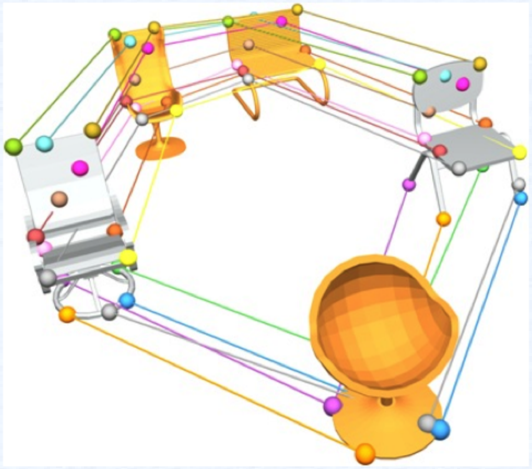


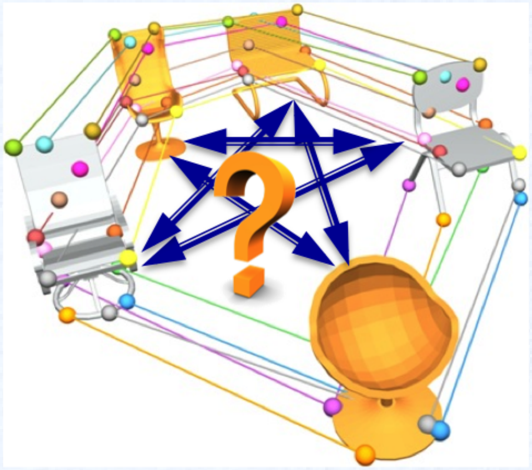
Input Maps

Challenge 3: Incomplete Input

- **Partial Input Matches**
  - pairwise matching across all object pairs is

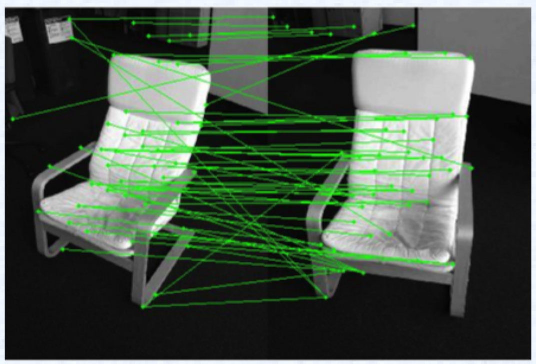
    — *computationally expensive*
    — *sometimes inadmissible*



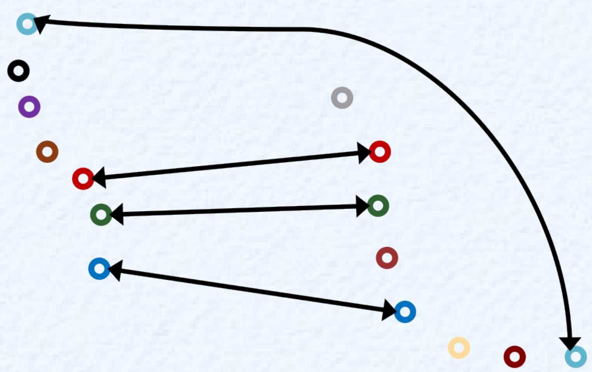


Our Goal

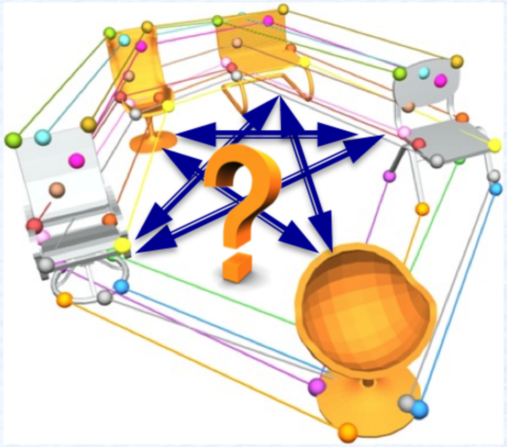
- **Develop an effective joint recovery method**
  - strong theoretical guarantee (*address the 3 challenges*)
  - parameter free
  - computationally feasible



tolerate dense errors



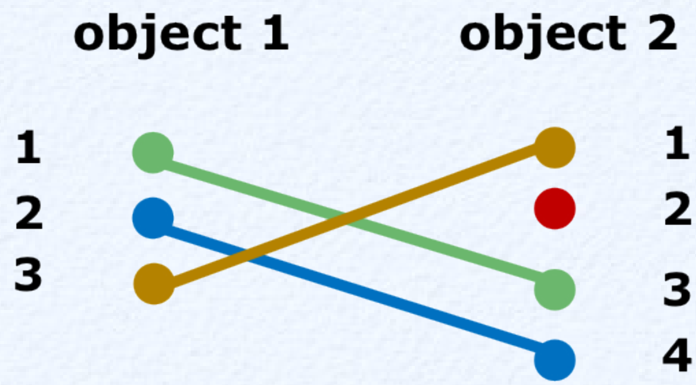
handle partial similarity



fill in missing matches

(Partial) Maps

- One-to-one maps between (sub)-sets of elements



  - subgraph matching / isomophism

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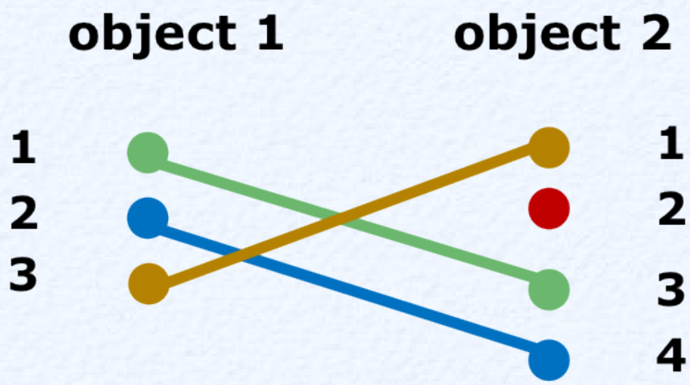
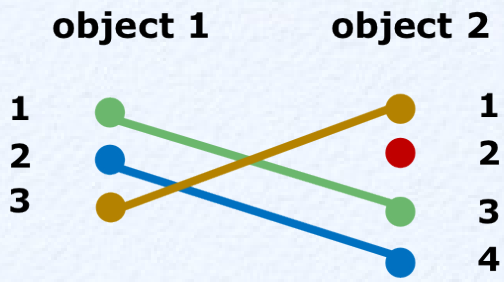


  - subgraph matching / isomophism

- Encode the maps across 2 objects by a 0-1 matrix

$$\boldsymbol{X}_{12} := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Matrix Representation



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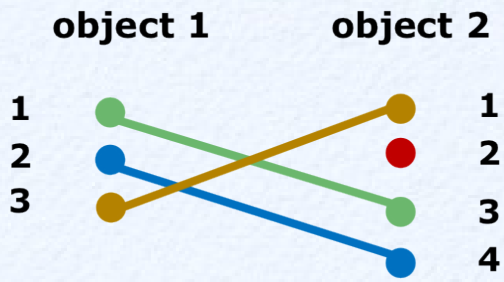


$$X_{12} := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

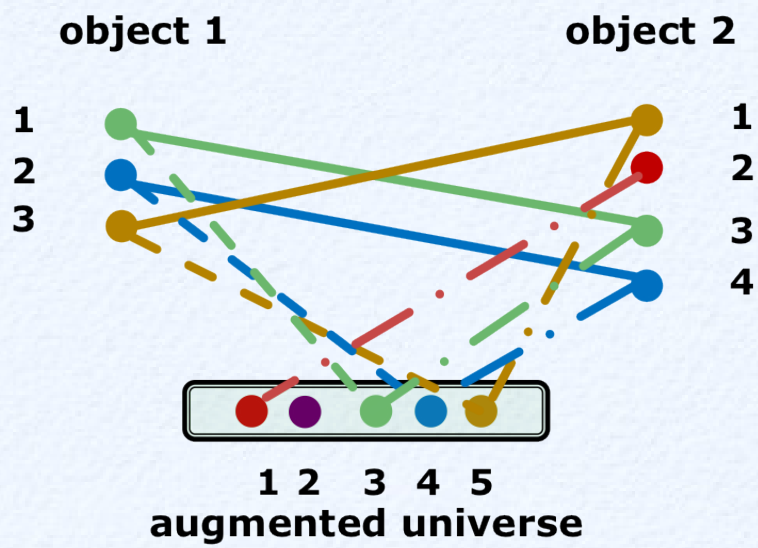
- Consider n objects
- Matrix representation for a collection of maps

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X}_{12} & \cdots & \boldsymbol{X}_{1n} \\ \boldsymbol{X}_{21} & \boldsymbol{I} & \cdots & \boldsymbol{X}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{n1} & \boldsymbol{X}_{n2} & \cdots & \boldsymbol{I} \end{bmatrix}$$

  - Diagonal blocks: identity matrices (self-isomophism)
  - Sparse

Alternative Representation: Augmented Universe

- All objects / sets are sub-sampled from the same universe (of size m).



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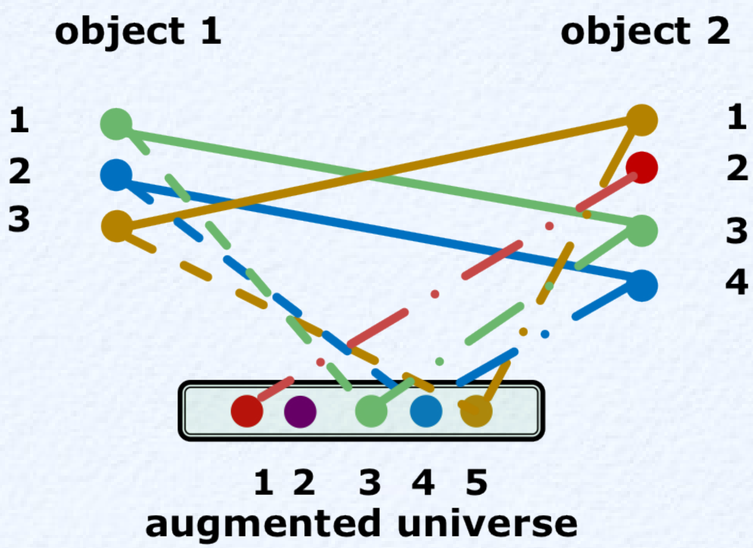

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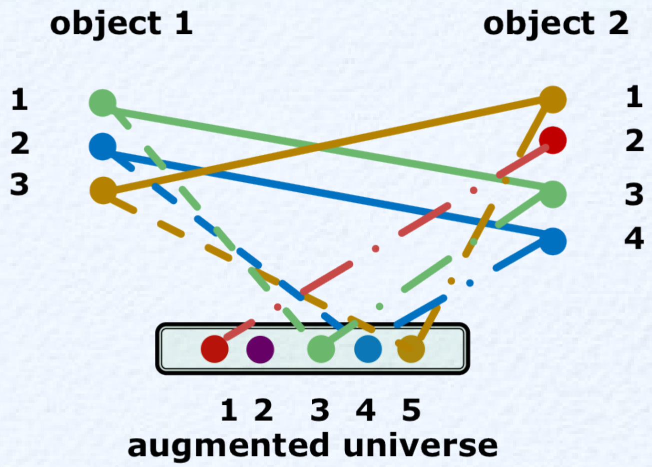
- Map matrix $\boldsymbol{Y}_i$ between object i and the universe

$$\boldsymbol{Y}_1 := \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{m \text{ columns}}, \quad \boldsymbol{Y}_2 := \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{m \text{ columns}} \quad \Rightarrow \quad \boldsymbol{X}_{12} = \boldsymbol{Y}_1 \boldsymbol{Y}_2^\top$$

P.S.D. and Low-Rank Structure



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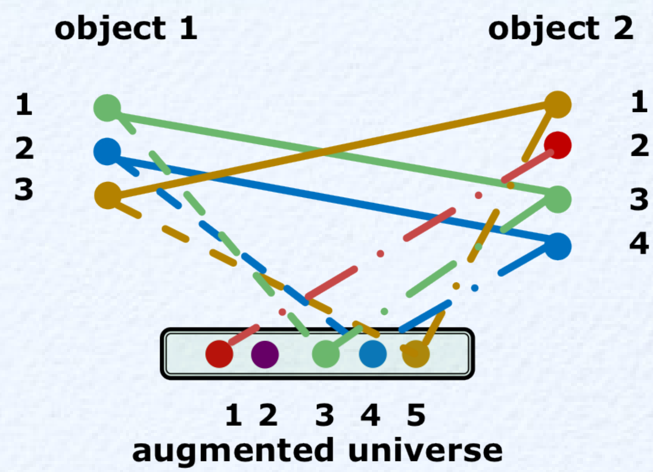

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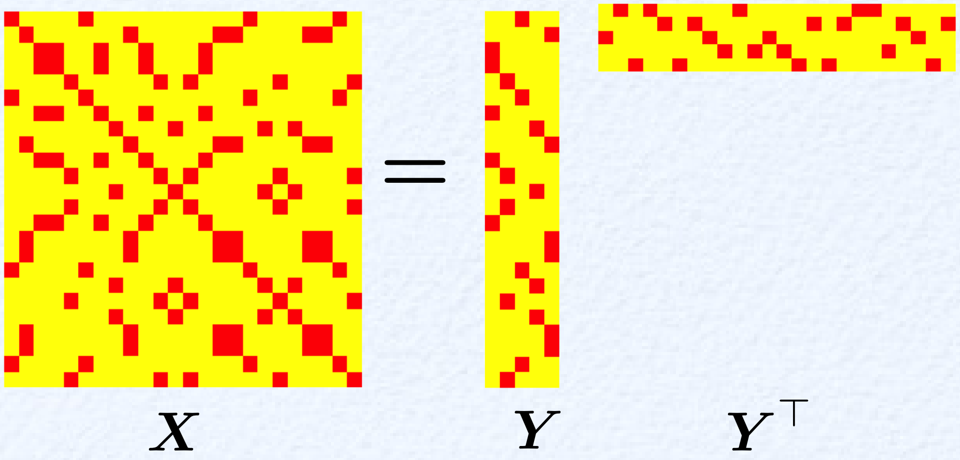
- **positive semidefinite and low rank:** $\text{rank}(\boldsymbol{X}) \le m.$
  - m: universe size

Summary of Matrix Structure

A consistent map matrix X

1. $\boldsymbol{X} \succeq 0$
2. low-rank
3. sparse (0-1 matrix)
4. $\boldsymbol{X}_{ii} = \boldsymbol{I}$



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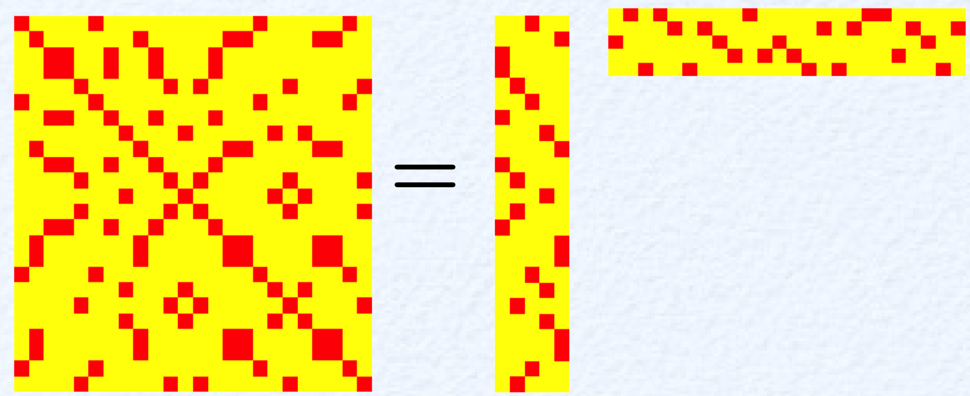
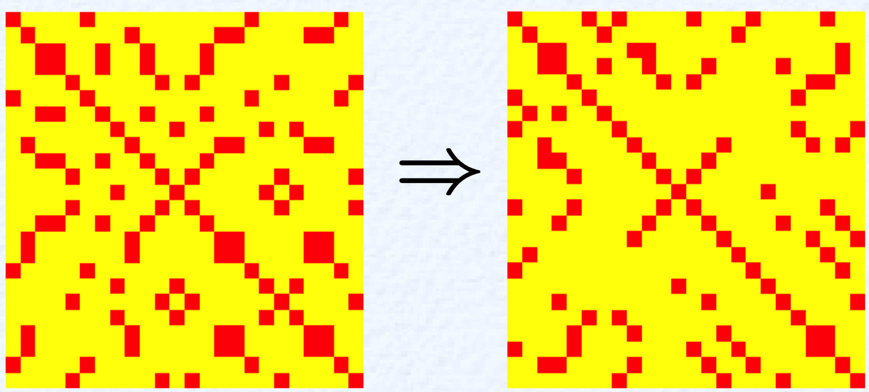


$\boldsymbol{X}$ $\quad$ $\boldsymbol{Y}$ $\quad$ $\boldsymbol{Y}^\top$



ground truth $\boldsymbol{X}$ $\quad$ input maps $\boldsymbol{X}^{\text{in}}$

Input map matrix X^{in}

- a noisy version of $\boldsymbol{X}$
  — *input errors*
- missing entries
  — *incomplete inputs*

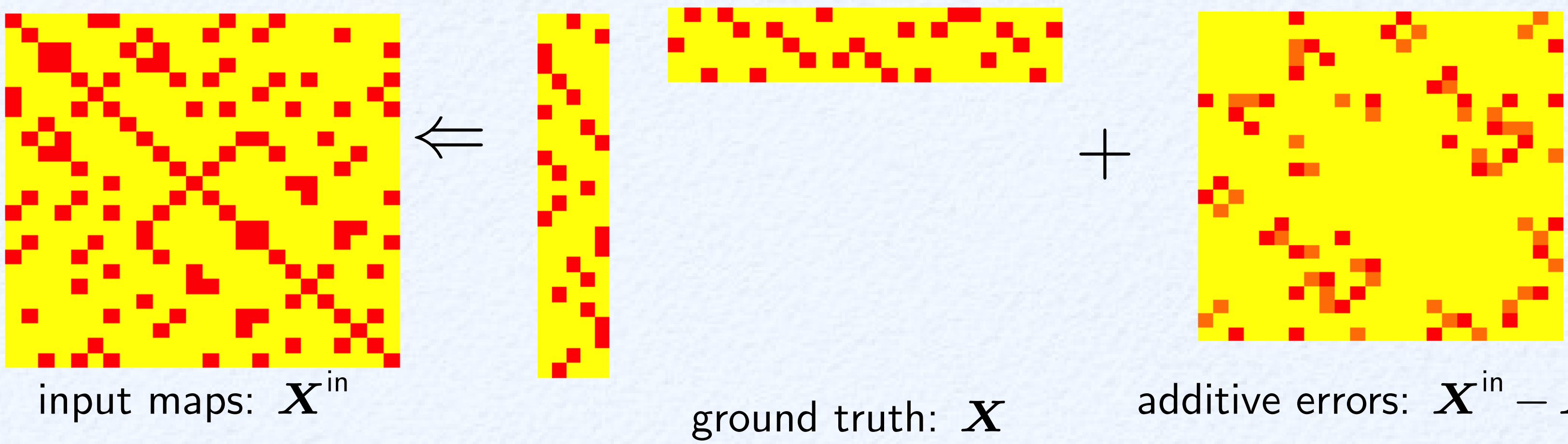
Low Rank + Sparse Matrix Separation?

input maps: $\boldsymbol{X}^{\text{in}}$ $\Leftarrow$ ground truth: $\boldsymbol{X}$ $+$ additive errors: $\boldsymbol{X}^{\text{in}} - \boldsymbol{X}$

- **Robust PCA / Matrix Completion?**
  - Candes et al
  - Chandrasekahran et al

$$\text{minimize}_{\boldsymbol{L},\boldsymbol{S}} \quad \underset{\text{(low rank)}}{\|\boldsymbol{L}\|_*} + \underset{\text{(sparse)}}{\|\boldsymbol{S}\|_1}, \quad \text{s.t.} \quad \boldsymbol{X}_{\text{in}} = \underset{\substack{\Downarrow \\ \text{estimate of } \boldsymbol{X}}}{\boldsymbol{L}} + \boldsymbol{S}$$

Outlier Component is Highly Biased



- **Robust PCA can handle dense corruption if**
  - the sparse component exhibits **random sign patterns**

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- **Our Case?**

$$\mathbb{E}\left[\boldsymbol{X}^{\text{in}} - \boldsymbol{X}\right] = p_{\text{true}}\boldsymbol{X} + \underbrace{(1 - p_{\text{true}})}_{\text{corruption rate}} \cdot \frac{1}{m}\mathbf{1}\cdot\mathbf{1}^{\top} - \boldsymbol{X} = \underbrace{(1 - p_{\text{true}})\left(\frac{1}{m}\mathbf{1}\cdot\mathbf{1}^{\top} - \boldsymbol{X}\right)}_{\text{highly biased}}$$

spectral norm: $(1 - p_{\text{true}})\, n$

Debias the Error Components

Original Form

$$\boldsymbol{X} := \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ \vdots \\ \boldsymbol{Y}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_1^\top & \boldsymbol{Y}_2^\top & \cdots & \boldsymbol{Y}_n^\top \end{bmatrix} \succeq 0$$

Augmented Form

$$\begin{bmatrix} m & \boldsymbol{1}^\top \\ \boldsymbol{1} & \boldsymbol{X} \end{bmatrix} := \begin{bmatrix} \boldsymbol{1}^\top \\ \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ \vdots \\ \boldsymbol{Y}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{Y}_1^\top & \cdots & \boldsymbol{Y}_n^\top \end{bmatrix} \succeq 0$$

- Equivalently,

$$\boldsymbol{X} - \underbrace{\frac{1}{m}\boldsymbol{1}\boldsymbol{1}^\top}_{\text{debiasing}} \succeq 0$$

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- Equivalently,

$$\boldsymbol{X} - \underbrace{\frac{1}{m}\mathbf{1}\mathbf{1}^\top}_{\text{debiasing}} \succeq 0$$

- rank $\left(\boldsymbol{X} - \frac{1}{m}\mathbf{1}\mathbf{1}^\top\right) = \text{rank}(\boldsymbol{X}) - 1 \quad \Rightarrow \quad$ one more degree of freedom

Objective Function

$$\boldsymbol{X} \geq \mathbf{0}, \quad \boldsymbol{X} \succeq \mathbf{0}$$

- Ecourage consistency with provided maps

$$\langle \boldsymbol{X}, \boldsymbol{X}^{\mathsf{in}} \rangle \quad \text{(to maximize)}$$

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- Promote Sparsity

$$\|\boldsymbol{X}\|_1 = \langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^\top \rangle \quad \text{(to minimize)}$$

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$$\|\boldsymbol{X}\|_1 = \langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^\top \rangle \quad \text{(to minimize)}$$

- Encourage Low-Rank Structure?
  - minimize nuclear norm? – not necessary ($\|\boldsymbol{X}\|_*$ is fixed)

Objective Function

$$\boldsymbol{X} \geq \mathbf{0}, \quad \boldsymbol{X} \succeq \mathbf{0}$$

- Ecourage consistency with provided maps

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- Encourage Low-Rank Structure?
  - minimize nuclear norm? – not necessary ($\|\boldsymbol{X}\|_*$ is fixed)

Objective Function (to minimize)

$$f(\boldsymbol{X}) := -\langle \boldsymbol{X}, \boldsymbol{X}^{\text{in}} \rangle + \lambda \langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^\top \rangle$$

MatchLift: tractable convex program

MatchLift

$$\begin{aligned} \text{minimize}_{\boldsymbol{X}} \quad & -\left\langle \boldsymbol{X}, \boldsymbol{X}^{\mathsf{in}} \right\rangle + \lambda \left\langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^\top \right\rangle \\ \text{subject to} \quad & \boldsymbol{X} \geq \mathbf{0}, \\ & \begin{bmatrix} m & \mathbf{1}^\top \\ \mathbf{1} & \boldsymbol{X} \end{bmatrix} \succeq \mathbf{0}, \\ & \boldsymbol{X}_{ii} = \boldsymbol{I}. \end{aligned}$$

- Efficient Semidefinite Program

MatchLift: tractable convex program

MatchLift

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$$

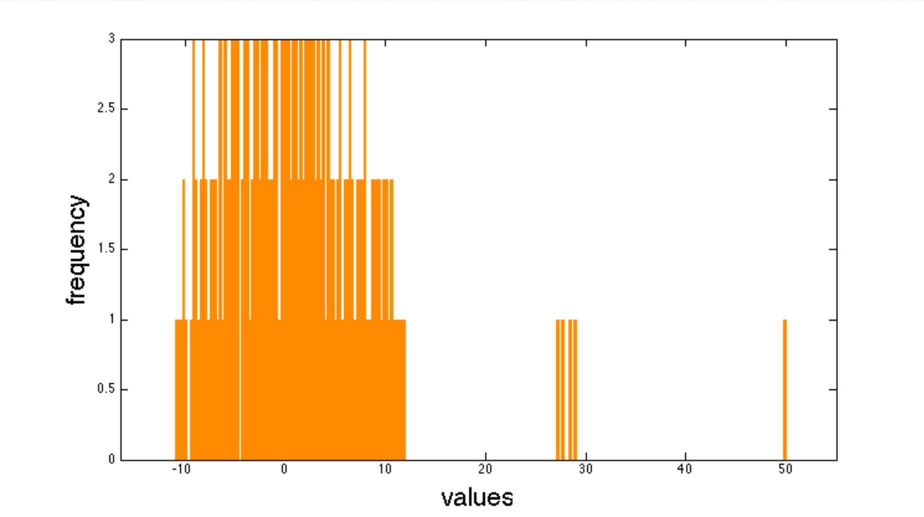
- Efficient Semidefinite Program
- Caveat: m is usually unkonwn!

Pre-Estimate m: Spectral Method

Spectral Method

1. Trim $\boldsymbol{X}^{\text{in}}$
2. $m \quad \longleftarrow \quad \#$ dominant eigenvalues of $\boldsymbol{X}^{\text{in}}$

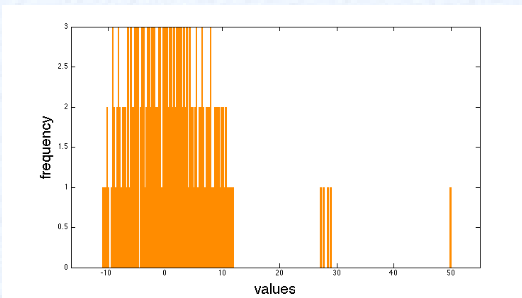
- The eigenvalues λ_i experience a sharp decrease around λ_m



$$n = 50, m = 5$$

Two-Step Procedure: MatchLift

1. **Pre-Estimate m:**



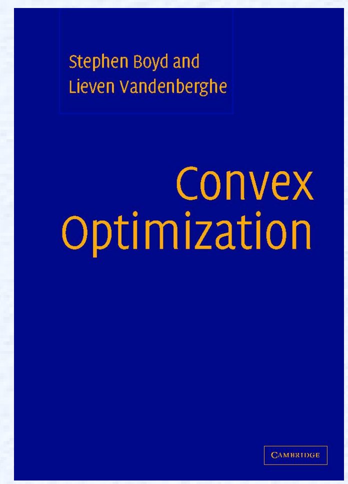
Spectral Method

1. Trim $\boldsymbol{X}^{\text{in}}$
2. $\boldsymbol{m} \quad \longleftarrow \quad$ # dominant eigenvalues of $\boldsymbol{X}^{\text{in}}$

2. **Joint Matching via Convex Relaxation:**

Convex Programming

$$
\begin{aligned}
\text{minimize}_{\boldsymbol{X}} \quad & -\left\langle \boldsymbol{X}, \boldsymbol{X}^{\text{in}} \right\rangle + \lambda \left\langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^{\top} \right\rangle \\
\text{subject to} \quad & \boldsymbol{X} \geq \mathbf{0}, \\
& \begin{bmatrix} \boldsymbol{m} & \mathbf{1}^{\top} \\ \mathbf{1} & \boldsymbol{X} \end{bmatrix} \succeq \mathbf{0}, \\
& \boldsymbol{X}_{ii} = \boldsymbol{I}.
\end{aligned}
$$



Exact Recovery via MatchLift

- **Randomized Model**: $\boldsymbol{n}$ objects, universe size $\boldsymbol{m}$
  - Each object contains a fraction $\underbrace{\boldsymbol{p}_{\text{set}}}_{\text{undersampling factor: partial similarity}}$ of m elements

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Theorem (ChenGuibasHuang'14). MatchLift with $\boldsymbol{\lambda} = \sqrt{\boldsymbol{p}_{\text{obs}}}$ is exact with high probability if

$$p_{\text{true}} \gtrsim \frac{\log^2(mn)}{p_{\text{set}}^2 \sqrt{p_{\text{obs}}} n}$$

Exact Recovery via MatchLift

$$\text{minimize}_{\boldsymbol{X}} \quad -\langle \boldsymbol{X}, \boldsymbol{X}^{\text{in}} \rangle + \lambda \langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^{\top} \rangle, \qquad \text{s.t. feasible}$$

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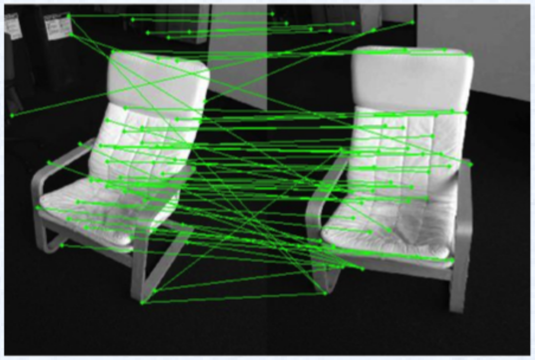

- **Dense Error Correction**

$$\text{error correction ability} \approx 1 - 1/\sqrt{n}$$

when p_{set} and p_{obs} are constants.

Exact Recovery via MatchLift

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- **Incomplete Input Matches**
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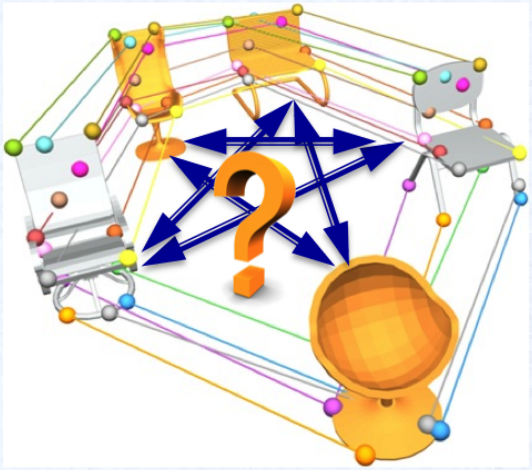

Exact Recovery via MatchLift

$$\text{minimize}_{\boldsymbol{X}} \quad -\langle \boldsymbol{X}, \boldsymbol{X}^{\text{in}} \rangle + \lambda \langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^\top \rangle, \qquad \text{s.t. feasible}$$

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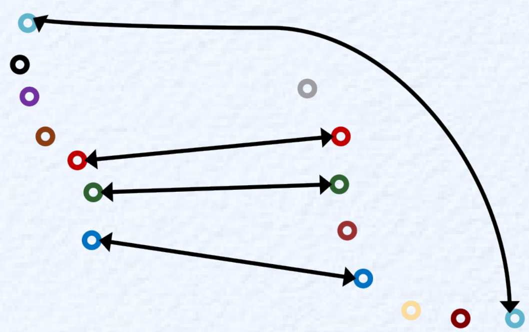
- **Incomplete Input Matches**
  - *Error correction ability decays at rate* $1/\sqrt{p_{obs}}$
- **Partial Similarity**
  - *Error correction ability decays at rate* $1/p_{set}^2$



Optimality of MatchLift

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- Is MatchLift Optimal?

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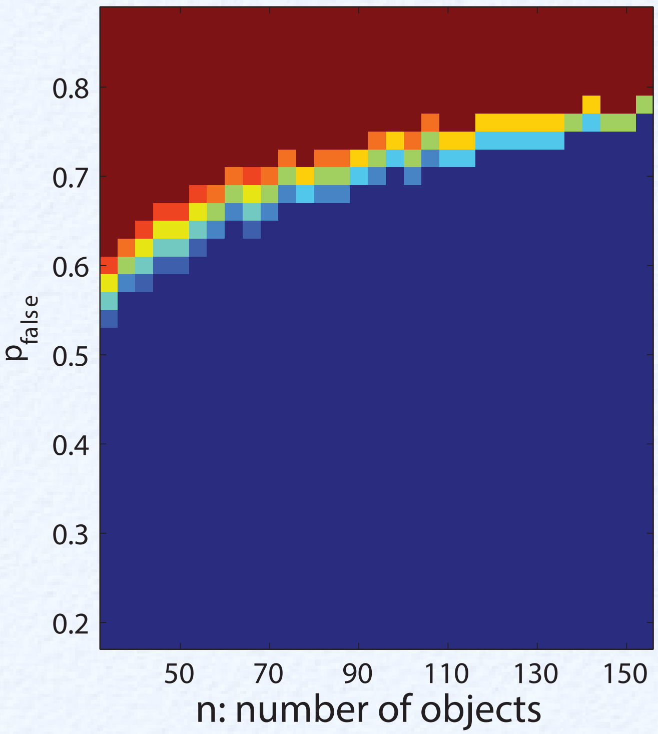
- Is MatchLift Optimal?
- Information Theoretic Limits under Random Measurement Graphs
  - *Fano's inequality*

Theorem (ChenGoldsmith'14). If the universe size m is a constant, then

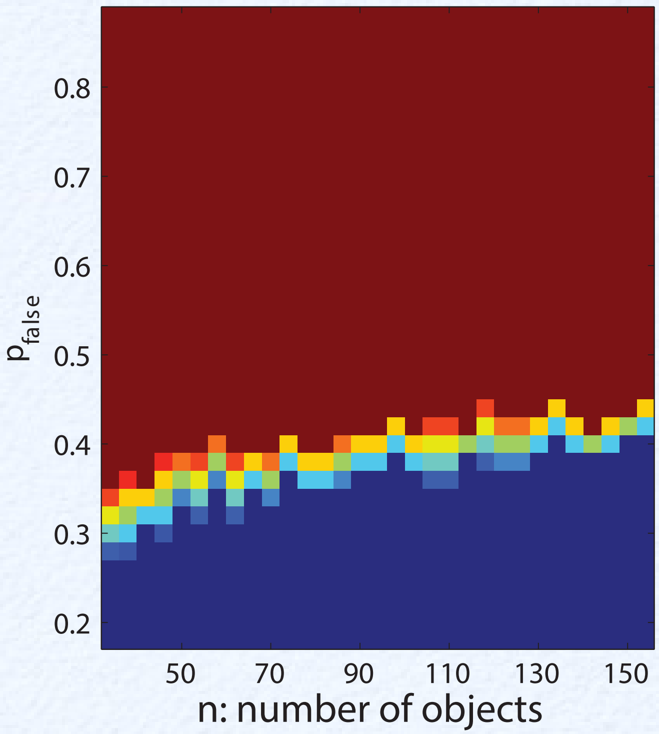
$$\text{No method works if} \quad p_{\text{true}} \lesssim \frac{1}{\sqrt{p_{\text{obs}} n}} \left(\approx \frac{1}{\sqrt{\text{avg-degree}}}\right)$$

Phase Transitions in Empirical Success Probability

- **Synthetic Data (input error rate v.s. # objects)**



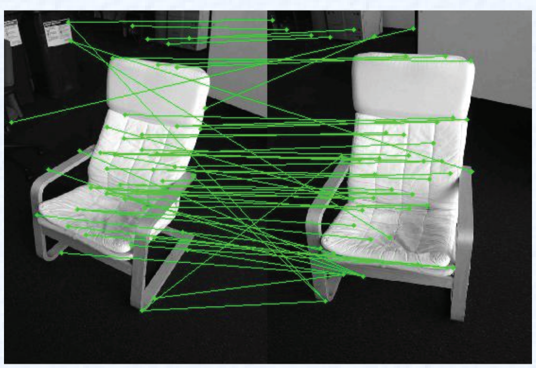
MatchLift



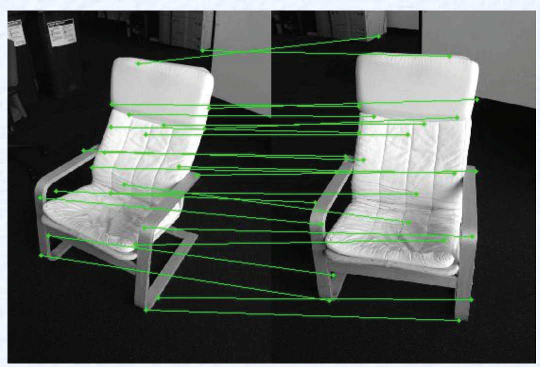
Robust PCA

Benchmark: Chairs

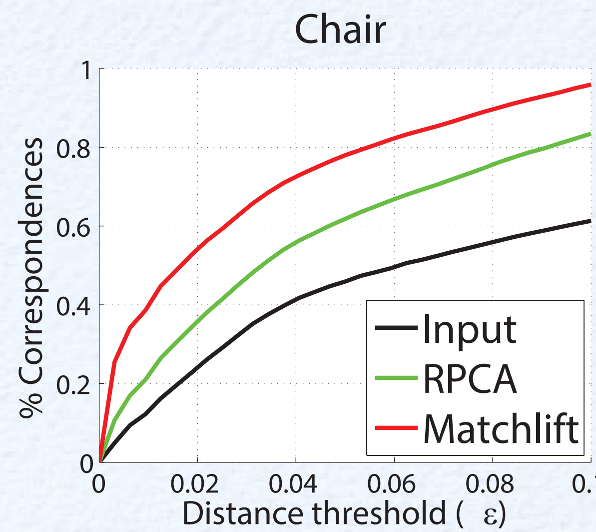
benchmark



initial maps

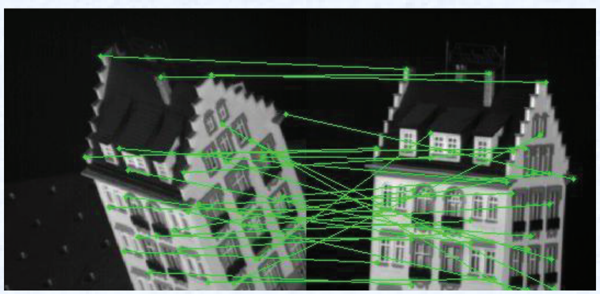


optimized maps



Benchmark: CMU Hotel

benchmark



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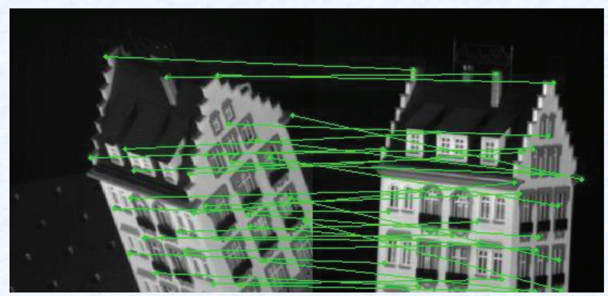

optimized maps

Input	MatchLift	RPCA	Leordeanu et al. 12
64.1%	100%	90.1%	94.8%

Concluding Remarks

- **MatchLift**
  - Dense error correction (near-optimal when m is constant)
  - Allow partial similarity
  - Incomplete inputs

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- **MatchLift**
  - Dense error correction (near-optimal when m is constant)
  - Allow partial similarity
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- **Future direction**
  - Pairwise matching and joint refinement all at once
  - More scalable algorithm
    - e.g. via non-convex optimization?

Paper and Code

- **Near-Optimal Joint Object Matching via Convex Relaxation**
  - Yuxin Chen, Leonidas J. Guibas, and Qixing Huang
    - *International Conference on Machine Learning (ICML), 2014*
  - **Arxiv:** `http://arxiv.org/abs/1402.1473`
  - **Code:** `http://web.stanford.edu/~yxchen/codes/code_MatchLift.zip`

Thank You! Questions?