An Upper Bound on Multi-hop Transmission Capacity with Dynamic Routing Selection

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Abstract—This paper develops an upper bound on the end-to-end transmission capacity of multi-hop wireless networks, in which all nodes are randomly distributed. Potential source-destination paths are dynamically selected from a pool of randomly located relays, from which a closed-form bound on the outage probability is derived in terms of the number of potential paths. This in turn gives an upper bound on the number of successful transmissions that can occur per unit area, which is known as the transmission capacity. The upper bound results from assuming independence among the potential paths, and can be viewed as the maximum diversity case. A useful aspect of the upper bound is its simple form for an arbitrary-sized network, which allows us to immediately observe how the number of hops and other network traits affect spatial throughput. Our analysis indicates that predetermined routing approach (such as nearest-neighbor) cannot achieve optimal throughput: more hops are not necessarily helpful in interference-limited networks compared with single-hop direct transmission.

I. Introduction

In a distributed wireless network with random node locations, determining the precise network capacity is a long-standing open problem that includes many other simpler open problems as special cases. Therefore, suboptimal analytical approaches that provide insight into the achievable throughput and inform improved protocol design are well-motivated, even if they fall short of strict upper bounds. Multihop routing is generally considered necessary in large wireless networks, both to insure connectivity and to improve the throughput via spatial reuse and diversity (cooperation) gains. In this paper we explore optimal multihop strategies by considering dynamic path selection. Pre-determined routing strategies such as nearest-neighbor routing, although they may perform fairly well on average, are generally not optimal for a given network state (which includes node positions and all the channels between them). In this paper, we are interested in how the inherent randomness in the network can be better harvested to improve the end-to-end success probability and hence throughput over more static approaches.

A. Related Work and Motivation

The best-known metric for studying end-to-end network capacity is the transport capacity approach pioneered by [1] and extended by numerous other researchers to more general operating regimes, e.g. [2], [3], [4]. In these works, nearest neighbor multihop routing has been shown to be order optimal

in the power-limited regime, while hopping across clusters with distributed MIMO can achieve order-optimal throughput in bandwidth-limited and power-inefficient regimes. However, most of these results only can be proven to hold for asymptotically large networks. Separately, multihop capacity can be studied in a line network without explicitly considering additional interference [5], [6]. This approach is helpful in comparing the impact of additional hops in bandwidth and power-limited networks, but fails to account for the interference inherent in a large wireless network.

If node locations are modelled as a homogeneous Poisson point process (HPPP), a number of convenient results can be applied from stochastic geometry, e.g. [7], in particular to compute outage probability relative to an SINR threshold. These expressions can be inverted to give the maximum transmit intensity at a specified outage probability, which yields the transmission capacity of the network [8]. While the transmission capacity can often be expressed in closed-form without resorting to asymptotics, it is a single-hop or "snapshot" metric. Recent work [9], [10] has considered transmission capacity (and outage probability) with two-hop relay selection, but more general multi-hop routing has not proven tractable unless several other strong assumptions are made, e.g. that all relays are on a straight line and all outages are independent [11], [12]. The outage of a predetermined route does not preclude the possibility of successful communication over other routes. In fact, we will show that since a pool of randomly located relays with varying channels provides more potential routes, the more randomness in the network, the better.

B. Contributions

This work determines a closed-form upper bound on transmission capacity as a function of outage constraint ε and the number of relays m for a general class of multi-hop routing strategies. This is given in Corollary 2, which follows from Lemma 1 on the number of possible $\mathcal{S} - \mathcal{D}$ routes in the network, which is the main technical result in the paper. These results assume a general exponential form of success probability, which captures most commonly used channel models as special cases (including path loss, Rayleigh fading, and Nakagami).

Instead of predetermined routing, dynamic selection from random relay sets under varying channel states are taken into consideration in order to get diversity gain. We have also derived a lower bound on the end-to-end outage probability (Corollary 1), which can be expressed as an exponential function with respect to the expected number of potential paths.

This result implies that higher throughput can be achieved when the correlation between the states of different hops is low and hence randomness and opportunism is high. The basic approach is to map all relay combinations to a higher dimensional space and focusing on the level set with respect to the success probability function.

II. MODELS AND PRELIMINARIES

A. Models and Assumptions

We assume that the locations of all sources are a realization of HPPP Ξ_t of intensity λ_t, and a fixed-portion set of relays are also randomly deployed in the plane with homogeneous Poisson distribution independent of Ξ_t with spatial density $\frac{1-\gamma}{\gamma}\lambda_t$, where $\gamma \in (0,1)$ is a fixed constant. Suppose each session uses k hops with the assistance of the relays, and transmission rate $b \approx \log(1+\beta)$ is required for successful transmission, where β is therefore the required SINR. Since b is simply a constant function of β, we ignore it for simplicity. Thus, denoting by ε as the target end-to-end outage probability, transmission capacity [8] can be defined as:

$$C_m(\varepsilon) = (1-\varepsilon) \max_{\Pr(\mathrm{SIR}<\beta) \leq \varepsilon} \frac{\lambda_t}{k}. \tag{1}$$

This follows because each hop requires a time slot, so the overall throughput must be normalized by k. Here, only one transmitter per route can be active at a time. It should be noted that "pipelining" or intraroute spatial reuse does not change the capacity in a network perspective [11]. This primary metric characterizes the maximum end-to-end contention density thinned by the success probability $1-\varepsilon$, which determines the maximum expected throughput per unit area.

Suppose that all transmitters employ equal amounts of power, the network is interference-limited, and that every destination node is a distance R from the source node. Relays can be selected from all nodes in the feasible region. For point-to-point transmission from node i to node j at a distance r_{ij}, the requirement for successful reception is expressed in terms of signal-to-interference ratio (SIR) constraint:

$$\mathrm{SIR}_{ij} = \frac{\|h_{ij}\|^2 r_{ij}^{-\alpha}}{\sum_{k \neq i} \|h_{kj}\|^2 r_{kj}^{-\alpha}} \geq \beta \tag{2}$$

where α denotes the path loss exponent, and h_{ij} the i.i.d. fading factor experienced by the path from i to j.

If the locations of the interference nodes follow an HPPP, (2) often suggests an exact or approximate exponential form for single-hop success probability. That is, given that the packet is transmitted from node i to next hop receiver j over distance r_{ij} and spatial density λ_t, the success probability is given as:

$$g_0(r_{ij}, \lambda_t) = G \exp(-\lambda_t K r_{ij}^2), \tag{3}$$

where G and K are variables which depends on specific channel models and which are independent of r_{ij} and λ_t. This is true for Rayleigh fading, Nakagami fading, and path loss models without fading. We omit the detailed derivation here but note that for Rayleigh fading:

$$K_{RF} = 2\pi\beta^{\frac{2}{\alpha}}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}\right)/\alpha, \quad G_{RF} = 1; \tag{4}$$

with $\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\mathrm{d}t$ being the Gamma function; for path loss model:

$$K_{PL} = \pi\beta^{\frac{2}{\alpha}}, \quad G_{PL} = 1. \tag{5}$$

Moreover, G can be different from 1 in some regime with Nakagami fading. For the rest of paper, we leave the results in terms of the general form (3).

III. GENERAL TRANSMISSION CAPACITY ANALYSIS

A. General Outage Probability Analysis

Suppose that m relays are used by a typical $\mathcal{S} - \mathcal{D}$ pair. We will build a connection between the outage probability and the expected number of relay sets that can connect the source and destination. Suppose that there is a transmission pair with source and destination located at $(-R/2, 0)$ and $(R/2, 0)$, respectively. Conditional on this typical pair, the spatial point process is still a HPPP with the same statistics. With the $i^{th}(1 \leq i \leq m)$ relay located at (x_i, y_i), let $Z_m = (x_1, y_1, \ldots, x_m, y_m)$ denote the location of this specific relay set. From Slivnyak's theorem, conditional on a typical transmission pair or finite number of nodes, the rest of the point process is still a homogeneous Poisson with the same density. Therefore, all relay combinations form a homogeneous point process in a $2m$-dimensional space $\mathcal{R}^{2m}$. We assume the effective spatial density to be $\tilde{\lambda}$. Assume that each relay combination Z_m can successfully assist in communications between the end-to-end transmission pair with probability $g_m(Z_m, \lambda_t)$. If we call the relay set that can successfully complete forwarding a potential relay set, the expected number of potential relay sets in a hypercube B, denoted by N_B, can be expressed as:

$$\lim_{v_{2m}(B) \to 0} \mathbb{E}(N_B) = \tilde{\lambda}^m g_m(Z_m, \lambda_t) v_{2m}(B), \tag{6}$$

where $v_{2m}(B)$ denotes the Lebesgue measure of B. Let N_m be the number of existing potential relay sets with the assistance of m relays. It should be noted that $\mathbb{E}(N_m)$ characterizes the expected number of different routes that can successfully forward the packets for each $\mathcal{S} - \mathcal{D}$ pair. A larger $\mathbb{E}(N_m)$ typically gives lower outage, since more successful routes can be expected to exist. The following theorem provides an outage probability bound for all success probability function $g_m(\cdot)$.

Theorem 1. *Assume that all end-to-end transmissions are achieved via $m+1$ hops with m relays. The end-to-end outage probability $p_{out}^{(m)}$ for any $\mathcal{S} - \mathcal{D}$ pair can be lower bounded as:*

$$p_{out}^{(m)} \geq \exp(-\mathbb{E}(N_m)) \tag{7}$$

Proof: Let the high-dimensional feasible region $\mathcal{F}$ for relay sets be the allowable range to select relays from. Denote by $\mathcal{A}$ the event that there is *no* relay set within $\mathcal{F}$ that can successfully complete forwarding. Ignoring the edge effect, we attempt to approximately divide $\mathcal{F}$ into n disjoint hypercubes $\mathcal{F}_i(1 \leq i \leq n)$ each of equal volume. For sufficiently large n, this approximation is exact. Let $\mathcal{A}_i(1 \leq i \leq n)$ be the event that there exists *no* potential relay set within $\mathcal{F}_i$ that can complete forwarding. Hence, $\mathcal{A} = \bigcap_{i=1}^n \mathcal{A}_i$. Consider two realization of

higher-dimensional point process ω and ω', and let $\omega \preceq \omega'$ if ω' can be obtained from ω by adding points. An event $\mathcal{A}_i$ is said to be decreasing if for every $\omega \preceq \omega'$, $\mathbb{I}_{\mathcal{A}_i}(\omega) \leq \mathbb{I}_{\mathcal{A}_i}(\omega')$ with $\mathbb{I}_{\mathcal{A}_i}$ denoting the indicator function of $\mathcal{A}_i$, it can be noted that $\mathcal{A}_i (1 \leq i \leq n)$ are all decreasing events. The Harris-FKG inequality [13] yields

$$\mathbb{P}(\mathcal{A}) = \mathbb{P}\left(\bigcap_{i=1}^{n} \mathcal{A}_i\right) \geq \prod_{i=1}^{n} \mathbb{P}(\mathcal{A}_i). \quad (8)$$

Consider the hypercube $\mathcal{F}_i$ as $[(x_1, y_1, \ldots, x_m, y_m), (x_1 + \delta x_1, y_1 + \delta y_1, \ldots, x_m + \delta x_m, y_m + \delta y_m)]$ when $\delta x_i \to 0, \delta y_i \to 0$. Define $Z_i = (x_1, y_1, \ldots, x_m, y_m)$. We can approximate the void probability if the Lebesgue measure $v_{2m}(\mathcal{F}_i)$ is small:

$$\mathbb{P}(\mathcal{A}_i) \approx 1 - g_m(Z_i, \lambda_t) \prod_{i=1}^{m}\left(1 - \exp(-\tilde{\lambda} \delta x_i \delta y_i)\right)$$
$$\approx \exp\left(-\tilde{\lambda}^m g_m(Z_i, \lambda_t) v_{2m}(\mathcal{F}_i)\right) \quad (9)$$

Let $n \to \infty$, then we can get the following lower bound

$$\mathbb{P}(\mathcal{A}) \geq \lim_{n \to \infty} \prod_{i=1}^{n} \mathbb{P}(\mathcal{A}_i)$$
$$= \exp\left(-\tilde{\lambda}^m g_m(Z_i, \lambda_t) \lim_{n \to \infty} \sum_{i=1}^{n} v_{2m}(\mathcal{F}_i)\right)$$
$$= \exp(-\mathbb{E}(N_m)) \quad (10)$$

■

It is immediate to find that the lower bound can only be obtained when all potential relay sets forms a Poisson point process, i.e., all potential points are independently scattered in the high-dimensional region. This indicates that lower correlation between different possible routes reduces the outage probability in essence by maximizing diversity. We can expect that this bound is tight and reasonable for small m (e.g. the bound is exact for single relay case) but may be loose for large m. This is because for a fixed pool of relays, the correlation between different routes increases when the number of relays m grows, i.e. for large m, many possible routes may share a couple of links. The expected number of different routes $\mathbb{E}[N_m]$ will be exactly calculated in the following subsection.

B. A Simple Upper Bound on Transmission Capacity

Now we begin to concentrate on the success probability of exponential forms. If a specific route is selected for packet delivery over m relays with hop distances $r_1, r_2, \ldots, r_{m+1}$, respectively, the probability for successful reception can be found as the product of each hop's success probability:

$$\prod_{i=1}^{m+1} g_0(r_i, \lambda_t) = G^{m+1} \exp\left(-\lambda_t K \sum_{i=1}^{m+1} r_i^2\right).$$

Here, we assume independence among each hop. This is reasonable because orthogonality among different subslots typically guarantees low correlation among different hops.

Suppose that the source and destination nodes are located at $(-R/2, 0)$ and $(R/2, 0)$. In the m relay case with the $i^{th} (1 \leq i \leq m)$ relay located at (x_i, y_i), let $Z_m = $

$(x_1, y_1, \ldots, x_m, y_m)$ denote the locations of the specific relay set, then we can define the corresponding distance statistics:

$$d_m(Z_m) \triangleq (x_1 + \frac{R}{2})^2 + \sum_{i=1}^{m-1}(x_i - x_{i+1})^2 + (x_m - \frac{R}{2})^2$$
$$+ y_1^2 + \sum_{i=1}^{m-1}(y_i - y_{i+1})^2 + y_m^2. \quad (11)$$

This is the sum of squares of hop distances. Hence, the routing success probability for a specific set of relays with location Z_m can be explicitly expressed as:

$$g_m(Z_m, \lambda_t) = G^{m+1} \exp(-\lambda_t K d_m(Z_m)). \quad (12)$$

For those relay sets with large $d_m(Z_m)$, the communication process becomes fragile and difficult to maintain due to low reception probability and large distance. Practical protocols usually search potential routes inside a locally finite area instead of from the infinite space. In order to leave the analysis general, we impose a constraint $d_m(Z_m) \leq D_m$ for the m relay case, where $D_m \to \infty$ reverts to the unconstrained case. A reasonably small constraint D_m is sufficient to achieve an aggregate rate arbitrarily close to capacity upper bound.

Besides, since only one transmitter is active at a time per route, each node is used as a relay by some $\mathcal{S} - \mathcal{D}$ pair with probability γ in each subslot. Thus, the pool of relays in each hop can be treated as the original HPPP Ξ thinned by probability γ. Hence, the location of all relay sets in $\mathcal{R}^{2m}$ is a realization of a point process with effective spatial density $\lambda^m(1 - \gamma)^m$. This leads to the following lemma.

Lemma 1. *If all end-to-end transmissions are achieved via $m + 1$ hops with the constraint $d_m(Z_m) \leq D_m$ $(Z_m \in \mathcal{R}^{2m})$, the expected number of potential relay sets can be given as:*

$$\mathbb{E}(N_m) = \frac{G^{m+1} \pi^m (1-\gamma)^m}{\gamma^m K^m (m+1)}\left\{\exp\left(-\frac{\lambda \gamma K R^2}{m+1}\right) - \exp\left(-\lambda \gamma K D_m\right) \sum_{0}^{m-1} \frac{1}{i!}\left(\lambda \gamma K\left(D_m - \frac{R^2}{m+1}\right)\right)^i\right\}.$$

Proof: The key point is that the isosurface of $d_m(Z_m)$ forms a high-dimensional elliptical surface. See Appendix. ■

This result indicates that larger m typically provides more diversity, because it provides more potential combinations of different relays, and the dynamically changing channel states provide more opportunities for a successful route. A larger feasible range for route selection D_m also increases the expectation, but since D_m mainly occurs in an exponentially vanishing term, a finite range is enough to approach the limits.

Corollary 1. *Assume that all end-to-end transmissions are achieved via $m + 1$ hops. If only a single transmission is allowed in each hop, the outage probability under the constraint $d_m(Z_m) \leq D_m$ $(Z_m \in \mathcal{R}^{2m})$ can be lower bounded as:*

$$p_{out}^{(m)} \geq \exp\left\{-\frac{G^{m+1} \pi^m (1-\gamma)^m}{\gamma^m K^m (m+1)}\left\{\exp\left(-\frac{\lambda \gamma K R^2}{m+1}\right) - \exp\left(-\lambda \gamma K D_m\right) \sum_{0}^{m-1} \frac{(\lambda \gamma K)^i}{i!}\left(D_m - \frac{R^2}{m+1}\right)^i\right\}\right\}. \quad (13)$$

This corollary provides a closed-form lower bound on the end-to-end outage probability without retransmissions. Especially for sufficiently large D_m, the lower bound reduces to:

$$p_{\text{out}}^{(m)} \geq \exp\left\{-\frac{G^{m+1}\pi^m(1-\gamma)^m}{\gamma^m K^m(m+1)}\exp\left(-\frac{\lambda\gamma KR^2}{m+1}\right)\right\},$$

which gives a clear characterization for "low-coherence" routing. As expected, multi-hop routing with the help of randomly deployed relays improves the success probability by providing diversity gain. Randomness in both relay locations and channel states brings more opportunities for us to exploit.

It should be noted that unlike the single hop scenario [8], our bound for outage probability is not globally monotone with λ if $D_m \nrightarrow \infty$. For sufficiently large D_m, Taylor expansion indicates that the outage probability is monotonically decreasing in $[0, \lambda_0]$ and increasing in $[\lambda_0, \infty]$, where

$$\lambda_0 = \frac{1}{\gamma K\left(D_m - \frac{R^2}{m+1}\right)}\ln\frac{(m+1)D_m}{R^2}. \qquad (14)$$

Taking the inverse over (λ_0, ∞) will yield the following bounds on maximum contention density.

Corollary 2. *If all end-to-end transmissions are achieved via $m + 1$ hops, the transmission capacity can be upper bounded as:*

$$\begin{aligned}
C_m(\varepsilon) &\leq \frac{m\ln\frac{G\pi(1-\gamma)}{K\gamma} + \ln G - \ln(m+1) - \ln\ln\frac{1}{\varepsilon}}{KR^2}(1-\varepsilon) \\
&\triangleq C_m^{ub}(\varepsilon),
\end{aligned} \qquad (15)$$

where $\varepsilon \geq \exp\left(-\frac{G^{m+1}\pi^m(1-\gamma)^m}{\gamma^m K^m(m+1)}\right)$.

Proof: Letting $D_m \to \infty$, and setting the outage probability equal to ε, (13) can be simplified as:

$$\varepsilon \geq \exp\left\{-\frac{G^{m+1}\pi^m(1-\gamma)^m}{\gamma^m K^m(m+1)}\exp\left(-\frac{\lambda\gamma}{m+1}KR^2\right)\right\}.$$

Notice that the effective spatial density is $\lambda\gamma/(m+1)$ and that ε is monotone in λ, we can immediately derive (15). Furthermore, in order to make sure $C_m(\varepsilon) > 0$, we obtain the constraint on ϵ in the corollary. ∎

Let $\overline{C_m^{ub}}(\varepsilon)$ be the bound under D_m constraint. When $D_m \nrightarrow \infty$ but is reasonably large, a simple approximation yields:

$$\overline{C_m^{ub}}(\varepsilon) + \Theta\{\exp\{-\overline{C_{ub}}(\varepsilon)K(m+1)D_m\}\} = C_m^{ub}(\varepsilon) \quad (16)$$

which implies the gap between the general bound and that with distance constraints will decay exponentially fast in D_m.

IV. Discussion

The analytical framework developed in this paper provides a simple closed-form upper bound to characterize the maximum achievable throughput with dynamic routing selection. Since $\Theta(\ln(m+1))$ is negligible compared with $\Theta(m)$ when m grows, the transmission capacity bound exhibits near linear scaling behavior with respect to the number of hops $m + 1$. This gain arises from the increasing diversity as m grows, because more hops means more choices of potentially successful routes. We caution that this gain is likely to become increasingly optimistic for large m, since longer potential routes will result in higher correlation that is not modeled, especially in a high-interference environment. Also, employing a large number of hops is likely to increase protocol overhead in practice quite substantially, which is not considered here. The proper choice of m under realistic correlation and protocol overhead models is an interesting topic for future research.

Another interesting result is that the transmission capacity bound is not sensitive to the outage constraint ε in the low outage regime, because the double logarithm as in $\ln\ln\frac{1}{\epsilon}$ largely reduces its sensitivity. This is quite different from single-hop transmission capacity, where the throughput exhibits linear scaling with ε [8]. Moreover, the Corollary 2 implies that increasing the portion of relay nodes will only logarithmically increase throughput. A large pool of relay sets is necessary to guarantee multi-hop selection, but bringing in too many relays may only provide limited throughput increment but incur substantial resource overhead. Additionally, the gap between constrained maximum density and unconstrained capacity bound is subject to exponential decay in $(m+1)D_m$. Hence, searching within a local and finite region is enough to approach the limit with a reasonable hop-distance constraint D_m.

This transmission capacity bound also makes clear that randomness and dynamic routing selection may be of significant importance. In fact, predetermined routing is unlikely to approach the throughput bound in interference-limited networks. A simple argument can be given as follows. Considering a typical $\mathcal{S} - \mathcal{D}$ pair, the outage probability can be bounded as:

$$1 - p_{\text{out}}^{(m)} = G\exp(-\lambda\gamma K d_m(Z_M)) \leq G\exp\left(-\frac{\lambda\gamma KR^2}{m+1}\right)$$

Equality can only be achieved if the m relays are equally spaced along the line segment between source and destination, which is unlikely to occur in a HPPP. Setting $\lambda\gamma(1-\varepsilon)/(m+1)$ to $C_m^{\text{ub}}(\varepsilon)$, we can immediately get an upper bound:

$$C_m^{\text{ub}}(\varepsilon) = \frac{1-\varepsilon}{KR^2}\ln\frac{G}{1-\varepsilon}, \qquad (17)$$

which is exactly equal to single hop capacity. This suggests that predetermined routing will not provide further throughput gain in interference-limited networks compared with single hop transmission. In fact, multihopping is primarily helpful in changing a power-limited network to an interference-limited one, consistent with [5], [11]. The design of transmission strategies exploiting the diversity gain are left for future work. We conjecture that hop-by-hop route selection – which is much more realistic in a distributed network than the complete route selection assumed here – will achieve a lower? the same? diversity order (and hence transmission capacity).

Appendix
Proof of Lemma 1

It can be noted that the isosurface of $d_m(Z_m) = a$ has the following coordinate geometry form:

$$X_{\text{sum}} + Y_{\text{sum}} = a, \qquad (18)$$

where $X_{\text{sum}} = (x_1 + \frac{R}{2})^2 + \sum_{i=1}^{m-1}(x_{i+1}-x_i)^2 + (x_m - \frac{R}{2})^2$ and $Y_{\text{sum}} = y_1^2 + \sum_{i=1}^{m-1}(y_{i+1}-y_i)^2 + y_m^2$.

If we treat x_i, y_i $(1 \le i \le m-1)$ as mutually orthogonal coordinates, then (18) forms a quadratic surface in $2m$-dimensional space. From the properties of quadratic forms, the x part and y part of (18) can be expressed as:

$$\begin{cases} X_{\text{sum}} = (\mathbf{CX}-\mathbf{R_x})^{\mathbf{T}}\mathbf{\Lambda_x}(\mathbf{CX}-\mathbf{R_x}) + t_m R^2, \\ Y_{\text{sum}} = (\tilde{\mathbf{C}}\mathbf{Y})^{\mathbf{T}}\mathbf{\Lambda_y}(\tilde{\mathbf{C}}\mathbf{Y}). \end{cases} \quad (19)$$

where $\mathbf{C}, \tilde{\mathbf{C}}$ are orthogonal matrices, $\mathbf{\Lambda_x}, \mathbf{\Lambda_y}$ are diagonal matrices, $\mathbf{R_x}$ has the explicit form of $[k_1, k_2, ..., k_{n-1}]^T R$, and t_m is a constant that will be determined in the sequel. Here, the translation of $\mathbf{X}$ ($\mathbf{Y}$) by $\mathbf{R_x}$ and the orthogonal transformation by $\mathbf{C}$ ($\tilde{\mathbf{C}}$) only results in rotation, or translation of the quadratic surface without changing the shape and size of it. Since the corresponding quadratic terms of X_{sum} and Y_{sum} have equivalent coefficients, we have $\mathbf{\Lambda_m} \triangleq \mathbf{\Lambda_x} = \mathbf{\Lambda_y}$. Denote the symmetric quadratic-form matrix with Y_{sum} as $\mathbf{A_m}$, then $\mathbf{A_m}$ is the following tridiagonal matrix of dimension m:

$$\mathbf{A_m} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}. \quad (20)$$

In fact, $\mathbf{\Lambda_m}$ is the canonical form of $\mathbf{A_m}$ with its eigenvalues on the main diagonal. Through transformation with $\mathbf{C}, \tilde{\mathbf{C}}$ and $\mathbf{R_x}$, X_{sum} and Y_{sum} can be brought to the explicit form

$$X_{\text{sum}} = \sum_{i=1}^{m} \lambda_i \tilde{x}_i^2 + t_m R^2, \quad (21)$$

$$Y_{\text{sum}} = \sum_{i=1}^{m} \lambda_i \tilde{y}_i^2. \quad (22)$$

where $\tilde{x}_i, \tilde{y}_i$ are the new orthogonal coordinates and λ_i is the ith eigenvalue of $\mathbf{A_m}$. By definition, X_{sum} is positive definite, and the following minimum value can be obtained if and only if m relays are placed equidistant along the line segment between the source and destination:

$$X_{\text{sum}} \ge \frac{\left(\left|\frac{R}{2}+x_1\right| + \cdots + \left|x_m - \frac{R}{2}\right|\right)^2}{m+1} \ge \frac{R^2}{m+1}. \quad (23)$$

Therefore, $t_m = \frac{1}{m+1}$. Now, (18) can be brought to:

$$\sum_{i=1}^{m} \lambda_i \tilde{x}_i^2 + \sum_{i=1}^{m} \lambda_i \tilde{y}_i^2 = a - \frac{R^2}{m+1}. \quad (24)$$

From the positive definiteness of $\mathbf{A_m}$, $\lambda_{\mathbf{i}} > 0$ for all i, i.e., (24) forms the surface of a $2m$-dimensional ellipsoid. The Lebesgue measure of the ellipsoid can be written as:

$$V_m(a) = \frac{\pi^m \left(a - \frac{R^2}{m+1}\right)^m}{m! \prod_{i=1}^{m} \lambda_i} = \frac{\pi^m \left(a - \frac{R^2}{m+1}\right)^m}{m! \det(\mathbf{A_m})}, \quad (25)$$

where $\det(\mathbf{A_m})$ can be computed by the Laplace expansion:

$$\det(\mathbf{A_m}) = 2\det(\mathbf{A_{m-1}}) - \det(\mathbf{A_{m-2}}). \quad (26)$$

Solving this recursive form with the initial value $\det(\mathbf{A_1}) = 2$ and $\det(\mathbf{A_2}) = 3$ yields:

$$\det(\mathbf{A_m}) = m+1 \quad \Rightarrow \quad V_m(a) = \frac{\pi^m(a - \frac{R^2}{m+1})^m}{(m+1)!}.$$

Define $B_\lambda = \lambda\gamma K$ and $H = \frac{\pi(1-\gamma)}{\gamma K}$. Integrating along different isosurface with $g(Z_m, \lambda\gamma) = \exp(-\lambda\gamma Ka)$ yields

$$\mathbb{E}(N_m)$$
$$= \int_{\frac{R^2}{m+1}}^{D_m} \lambda^m (1-\gamma)^m \frac{dV_m(a)}{da} G^{m+1} \exp(-B_\lambda a) da$$
$$= G^{m+1}\lambda^m(1-\gamma)^m \int_{\frac{R^2}{m+1}}^{D_m} \frac{m\pi^m \left(a - \frac{R^2}{m+1}\right)^{m-1}}{(m+1)!\exp(B_\lambda a)} da$$
$$= \frac{mG^{m+1}H^m \exp(-\frac{B_\lambda R^2}{m+1})}{(m+1)!} \int_0^{B_\lambda(D_m - \frac{R^2}{m+1})} \frac{x^{m-1}}{\exp(x)} dx$$
$$= \frac{G^{m+1}H^m}{m+1} \left\{ \exp\left(-\frac{B_\lambda R^2}{m+1}\right) - \right.$$
$$\left. \exp(-B_\lambda D_m) \sum_0^{m-1} \frac{1}{i!}\left(B_\lambda(D_m - \frac{R^2}{m+1})\right)^i \right\}. \quad (27)$$

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