Deflated HeteroPCA: Overcoming the curse of ill-conditioning in heteroskedastic PCA

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Abstract

This paper is concerned with estimating the column subspace of a low-rank matrix $\mathbf{X}^* \in \mathbb{R}^{n_1 \times n_2}$ from contaminated data. How to obtain optimal statistical accuracy while accommodating the widest range of signal-to-noise ratios (SNRs) becomes particularly challenging in the presence of heteroskedastic noise and unbalanced dimensionality (i.e., $n_2 \gg n_1$). While the state-of-the-art algorithm HeteroPCA emerges as a powerful solution for solving this problem, it suffers from "the curse of ill-conditioning," namely, its performance degrades as the condition number of \mathbf{X}^* grows. In order to overcome this critical issue without compromising the range of allowable SNRs, we propose a novel algorithm, called Deflated-HeteroPCA, that achieves near-optimal and condition-number-free theoretical guarantees in terms of both ℓ_2 and $\ell_{2,\infty}$ statistical accuracy. The proposed algorithm divides the spectrum of \mathbf{X}^* into wellconditioned and mutually well-separated subblocks, and applies HeteroPCA to conquer each subblock successively. Further, an application of our algorithm and theory to two canonical examples — the factor model and tensor PCA — leads to remarkable improvement for each application.

Keywords: principal component analysis (PCA), heteroskedastic noise, the curse of ill-conditioning, factor models, tensor PCA

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1 Introduction

In a diverse array of science and engineering applications, we are asked to identify a low-dimensional subspace that best captures the information underlying a large collection of high-dimensional data points, a classical problem that goes by the names of principal component analysis (PCA), subspace estimation, subspace tracking, among others (Johnstone and Paul, 2018; Balzano et al., 2018; Chen et al., 2021b). A simple yet useful mathematical model is of the following form: imagine we have an unknown large-dimensional matrix $X^* \in \mathbb{R}^{n_1 \times n_2}$ whose columns are high-dimensional vectors embedded in a *r*-dimensional subspace (so that X^* has rank $r \ll \min\{n_1, n_2\}$), and we seek to estimate the *column space* of X^* from noisy observations:

$$\boldsymbol{Y} = \boldsymbol{X}^{\star} + \boldsymbol{E} \in \mathbb{R}^{n_1 \times n_2},\tag{1}$$

where E stands for the noise matrix that contaminates the data. Despite decades-long research, there remain substantial challenges to handle heteroskedastic noise in high dimension, as we shall elaborate on below.

1.1 Challenges: unbalanced dimensionality and heteroskedasticity

How to achieve statistically efficient PCA in high dimension is an active research topic that has received much recent interest (Lounici, 2014; Johnstone and Paul, 2018; Cai et al., 2021; Zhu et al., 2019; Zhang et al., 2022; Agterberg et al., 2022). In this paper, we pay particular attention to the case where n_1 and n_2 are both enormous but highly *unbalanced* in the sense that $n_1 \ll n_2$, a scenario that arises frequently in, say, covariance estimation (when there are many noisy samples available) and tensor estimation (when one has to matrice the tensor before estimation). Such unbalanced dimensionality gives rise to unique challenges not present in the complement case: as the signal-to-noise ratio (SNR) keeps decreasing, one might soon enter a regime where consistent estimation of X^* is no longer infeasible but its column subspace — which is much smaller dimensional than the full matrix — remains estimatable. This regime is often considerably more challenging than the case with $n_2 = O(n_1)$, given that the majority of low-rank matrix estimation algorithms that directly attempt to estimate X^* become completely off.

One natural strategy that comes into mind is thus to estimate the column subspace of X^* by calculating the left singular subspace of the observed matrix Y (Cai and Zhang, 2018; Abbe et al., 2020; Chen et al., 2021b), which we shall refer to as the *vanilla SVD-based approach* throughout. In the case with $n_1 \ll n_2$, this simple scheme has only been shown to achieve the desired statistical performance when the noise matrix E is composed of i.i.d. entries, but falls short of effectiveness when handling *heteroskedastic* noise (i.e., the



Figure 1: Subspace estimation error vs. condition number κ of Σ^* . Here, we set $r = 2, n_1 = 200$ and $n_2 = 40,000$. The truth $X^* = U^* \Sigma^* V^{*\top}$ has rank 2 with $U^* \in \mathcal{R}^{n_1 \times 2}$ and $V^* \in \mathcal{R}^{n_2 \times 2}$ generated randomly. Plot (a) represents the noiseless case (E = 0). In Plot (b), we choose the two singular values of X^* as $\sigma_1^* = \kappa \sigma_2^*$ and $\sigma_2^* = 200$, generate $\{\omega_i\}_{1 \le i \le n_1}$ independently from Unif([0, 2]), and draw the entries of $E = [E_{i,j}]_{1 \le i \le n_1, 1 \le j \le n_2}$ independently such that $E_{i,j} \sim \mathcal{N}(0, \omega_i^2)$. We compare multiple subspace estimators here, where HeteroPCA is run with 100 iterations. For each estimator \hat{U} , we compute the spectral-normbased error $\|\hat{U}R_{\hat{U}} - U^*\|$ as κ varies, where $R_{\hat{U}} = \arg\min_{R \in \mathcal{O}^{r,r}} \|\hat{U}R - U^*\|_{\mathrm{F}}$; the results are averaged over 50 independent runs.

scenario where the variances of the entries of E are location-varying) (Zhang et al., 2022; Cai et al., 2021). This issue presents a hurdle to transferring this scheme from theory to practice, due to the ubiquity of heteroskedastic data in applications like social networks, recommendation systems, medical imaging, etc.

To mitigate this issue, at least two strategies have been proposed that attempt estimation by looking at the empirical covariance matrix (or gram matrix) YY^{\top} . Recognizing that large heteroskedastic noise might lead to significant bias in the diagonal of YY^{\top} that distorts estimation, one natural remedy is to zero out (or sometimes rescale) the diagonal entries of YY^{\top} before computing its eigendecomposition (Koltchinskii and Giné, 2000; Lounici, 2014; Florescu and Perkins, 2016; Loh and Wainwright, 2012; Montanari and Sun, 2018; Elsener and van de Geer, 2019; Cai et al., 2021; Ndaoud et al., 2021). A more refined iterative procedure called HeteroPCA was subsequently proposed by Zhang et al. (2022), which starts with the solution of diagonal-deleted PCA (cf. (10)) and alternates between:

- imputing the diagonal entries of $X^* X^{*\top}$;
- computing the rank-r eigenspace of YY^{\top} with its diagonal replaced by the imputed values.

See Section 3 for precise descriptions. In both theory and numerical experiments, this iterative paradigm yields enhanced performance compared to diagonal-deleted PCA (Zhang et al., 2022; Yan et al., 2021).

1.2 The curse of ill-conditioning

Nevertheless, one drawback stands out when running either diagonal-deleted PCA or HeteroPCA in practice; that is, both algorithms become ineffective as the condition number of X^* (when restricted to its non-zero singular values) grows. Let us illustrate this point more clearly via numerical experiments.

• (Numerical example) Consider the case where the unknown signal X^* has rank r = 2 and obeys $X^* = U^* \Sigma^* V^{*\top}$, where the columns of $U^* \in \mathbb{R}^{n_1 \times 2}$ (resp. $V^* \in \mathbb{R}^{n_1 \times 2}$) are the two left (resp. right) singular vectors of X^* , and $\Sigma^* \in \mathbb{R}^{2 \times 2}$ is a diagonal matrix composed of the two singular values $\sigma_1^* \geq \sigma_2^* > 0$ of X^* . Denote by $\kappa = \sigma_1^* / \sigma_2^*$ the condition number of Σ^* . We conduct a series of experiments based on randomly generated X^* with $n_2 \gg n_1$, as detailed in the caption of Figure 1. As illustrated in Figure 1, when κ is not too large, both diagonal-deleted PCA and HeteroPCA fail to return reliable estimates of the subspace U^* , even in the noiseless case (i.e., E = 0).

In summary, both diagonal-deleted PCA and HeteroPCA suffer from the "curse of ill-conditioning", namely, they might lead to grossly incorrect subspace estimates as the largest signal component strengthens with all other signal components unchanged. This observation is somewhat counter-intuitive; after all, altering the signal this way only serves to increase the SNR and hence simplify the task from the information-theoretic perspective. In this sense, the aforementioned curse of ill-conditioning seems to be algorithm-specific, although the two algorithms it concerns happen to be the state-of-the-art methods. All this naturally leads to the following question:

Can we overcome the above curse of ill-conditioning without compromising the advantages of both diagonal-deleted PCA and HeteroPCA?

1.3 This paper

As it turns out, we can answer the above question in the affirmative, which forms the main contribution of this paper. Our main findings are summarized as follows.

- Algorithm design. In an attempt to address the above question, we propose a new algorithm dubbed as Deflated-HeteroPCA on the basis of HeteroPCA. In a nutshell, the proposed algorithm divides the spectrum of X^* into well-conditioned yet mutually well-separated subblocks, and successively applies HeteroPCA to conquer each subblock. This approach counters the adverse influence of ill conditioning via successive "deflation" (a term borrowed from Dobriban and Owen (2019)), which gradually "deflates" the undesirable bias effect resulting from the diagonal deletion operation.
- Statistical guarantees. We develop sharp theoretical guarantees, in terms of both ℓ_2 (spectral-normbased) and $\ell_{2,\infty}$ estimation errors, for the proposed algorithm. Encouragingly, all of these statistical guarantees are condition-number-free, and match the minimax lower bounds established in Zhang et al. (2022) and Cai et al. (2021) (up to some logarithmic factors). To the best of our knowledge, these provide the first near-optimal results in the heteroskedastic PCA setting herein that (i) do not degrade as the condition number of the truth increases, and (ii) accommodate the widest range of SNRs.
- Consequences in two canonical examples. To illustrate the utility of our algorithm and theory, we develop concrete consequences of our results for two canonical examples: (a) the factor model, and (b) tensor PCA. We demonstrate that (i) Deflated-HeteroPCA achieves rate-optimal and condition-number-free estimation under the factor model, and (ii) Deflated-HeteroPCA followed by the HOOI algorithm improves upon the state-of-the-art performance guarantees for tensor PCA. Numerical experiments are carried out to corroborate the effectiveness of the propose algorithm.

Paper organization. The rest of the paper is organized as follows. We formulate the problem precisely in Section 2, and present the proposed algorithm in Section 3. The theoretical guarantees of our algorithm, along with their implications, are presented in Section 4. We develop concrete consequences of our results in two applications in Section 5. Additional numerical experiments are reported in Section 6, and a discussion of further related works is provided in Section 7. The technical proofs are collected in the Appendix.

1.4 Notation

Throughout this paper, we denote $[n] := \{1, \ldots, n\}$ for any positive integer n. We let bold capital letters (e.g., X) and bold lowercase letters (e.g., x) denote matrices and vectors, respectively. For any matrix $A \in \mathbb{R}^{n_1 \times n_2}$, $\lambda_i(A)$ and $\sigma_i(A)$ are used to represent the *i*-th largest eigenvalue (in magnitude) and the *i*-th largest singular value of A, respectively. Let $\|\cdot\|_{\mathrm{F}}$ indicate the Frobenious norm and $\|\cdot\|$ the spectral norm. We denote by $A_{i,:}$ and $A_{:,j}$ the *i*-th column and the *j*-th row of A, respectively. We also let $A_{:,i:j}$ denote the submatrix of A containing those columns with indices falling in [i, j]. Let $\|A\|_{2,\infty} := \max_i \|A_{i,:}\|_2$ denote the $\ell_{2,\infty}$ norm of A. We use $\mathcal{O}^{n,r} := \{U \in \mathbb{R}^{n \times r} : U^\top U = I_r\}$ to represent the set containing all $n \times r$ matrices with orthonormal columns. For any $U \in \mathcal{O}^{n,r}$, we define the projection matrix $\mathcal{P}_U = UU^\top$. Let $U_\perp \in \mathcal{O}^{n,n-r}$ denote the orthogonal complement of U. We use $\mathcal{P}_{\mathsf{diag}}(\cdot)$ to represent the projection operator that keeps all diagonal entries and sets to zero all non-diagonal entries; meanwhile, we define $\mathcal{P}_{\mathsf{off}-\mathsf{diag}}(M) := M - \mathcal{P}_{\mathsf{diag}}(M)$ for any $M \in \mathbb{R}^{n \times n}$. For any vector $a = (a_1, \ldots, a_n)$, we denote by $\mathsf{diag}(a) \in \mathbb{R}^{n \times n}$ the diagonal matrix

whose (i, i)-th entry is a_i . For any full-rank matrix $\boldsymbol{H} \in \mathbb{R}^{r \times r}$ with singular value decomposition (SVD) $\boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\top}$, we define the sign matrix

$$\operatorname{sgn}(\boldsymbol{H}) := \boldsymbol{U}\boldsymbol{V}^{\top}.$$
(2)

We let C, c, C_0, c_0, \ldots denote numerical constants whose values may change from line to line. The boldface calligraphic letters (e.g., \mathcal{X}) are used to represent tensors. For any tensor $\mathcal{G} \in \mathbb{R}^{r_1 \times r_2 \times r_3}$ and any matrix $V_1 \in \mathbb{R}^{n_1 \times r_1}$, we define the multi-linear product \times_1 as follows:

$$\boldsymbol{\mathcal{G}} \times_1 \boldsymbol{V}_1 = \left(\sum_{j=1}^{r_1} G_{j,i_2,i_3} V_{i_1,j}\right)_{i_1 \in [n_1], i_2 \in [r_2], i_3 \in [r_3]}$$

We can define \times_2 and \times_3 analogously. For any tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, let $\mathcal{M}_j(\mathcal{X}) \in \mathbb{R}^{n_j \times (n_1 n_2 n_3/n_j)}$ denote the *j*-th matricization of \mathcal{X} such that for any $(i_1, i_2, i_3) \in [n_1] \times [n_2] \times [n_3]$,

$$\left[\mathcal{M}_{1}\left(\boldsymbol{\mathcal{X}}\right)\right]_{i_{1},i_{2}+n_{2}(i_{3}-1)}=\left[\mathcal{M}_{2}\left(\boldsymbol{\mathcal{X}}\right)\right]_{i_{2},i_{3}+n_{3}(i_{1}-1)}=\left[\mathcal{M}_{3}\left(\boldsymbol{\mathcal{X}}\right)\right]_{i_{3},i_{1}+n_{1}(i_{2}-1)}=X_{i_{1},i_{2},i_{3}}.$$

The Frobenious norm of a tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is defined as

$$\|\boldsymbol{\mathcal{X}}\|_{\mathrm{F}} = \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} X_{i,j,k}^2\right)^{1/2}$$

The notation $f(n_1, n_2) \leq g(n_1, n_2)$ or $f(n_1, n_2) = O(g(n_1, n_2))$ means that $|f(n_1, n_2)| \leq Cg(n_1, n_2)$ holds for some numerical constant C > 0; we let $f(n_1, n_2) \geq g(n_1, n_2)$ indicate that $f(n_1, n_2) \geq C|g(n_1, n_2)|$ for some numerical constant C > 0; $f(n_1, n_2) \approx g(n_1, n_2)$ means that both $f(n_1, n_2) \leq g(n_1, n_2)$ and $f(n_1, n_2) \geq g(n_1, n_2)$ hold; we use the notation $f(n_1, n_2) \ll g(n_1, n_2)$ to represent that $f(n_1, n_2) \leq cg(n_1, n_2)$ holds for some sufficiently small constant c > 0, and we say $f(n_1, n_2) \gg g(n_1, n_2)$ if $g(n_1, n_2) \ll f(n_1, n_2)$. In addition, we use $f(n_1, n_2) = o(g(n_1, n_2))$ to indicate that $f(n_1, n_2)/g(n_1, n_2) \to 0$ as $\min\{n_1, n_2\} \to \infty$. For any $a, b \in \mathbb{R}$, we define $a \land b := \min\{a, b\}$ and $a \lor b := \max\{a, b\}$.

2 Problem formulation

Models and assumptions. Let us present a more precise description of the problem to be studied here. Imagine that we have access to the following noisy data matrix:

$$\boldsymbol{Y} = \boldsymbol{X}^{\star} + \boldsymbol{E} \in \mathbb{R}^{n_1 \times n_2},\tag{3}$$

where $\boldsymbol{E} = [E_{i,j}]_{1 \le i \le n_1, 1 \le j \le n_2}$ is a zero-mean noise matrix composed of independent entries, and $\boldsymbol{X}^{\star} = [X_{i,j}^{\star}]_{1 \le i \le n_1, 1 \le j \le n_2}$ is a rank-*r* matrix to be estimated. The SVD of the signal matrix \boldsymbol{X}^{\star} is given by

$$\boldsymbol{X}^{\star} = \boldsymbol{U}^{\star} \boldsymbol{\Sigma}^{\star} \boldsymbol{V}^{\star \top} = \sum_{i=1}^{r} \sigma_{i}^{\star} \boldsymbol{u}_{i}^{\star} \boldsymbol{v}_{i}^{\star \top} \in \mathbb{R}^{n_{1} \times n_{2}}.$$
(4)

Here, $\sigma_1^{\star} \geq \cdots \geq \sigma_r^{\star} > 0$ denote the singular values of X^{\star} , u_i^{\star} (resp. v_i^{\star}) represents the left (resp. right) singular vector associated with σ_i^{\star} , and we introduce the matrices $\Sigma^{\star} = \text{diag}(\sigma_1^{\star}, \ldots, \sigma_r^{\star}), U^{\star} = [u_1^{\star}, \ldots, u_r^{\star}] \in \mathcal{O}^{n_1, r}$ and $V^{\star} = [v_1^{\star}, \ldots, v_r^{\star}] \in \mathcal{O}^{n_2, r}$. Clearly, U^{\star} and V^{\star} represent the column and row subspaces of X^{\star} , respectively.

Moreover, we introduce additional definitions and assumptions to be used throughout.

• To begin with, let us introduce the following incoherence condition that appears frequently in the low-rank matrix estimation literature (Candès and Recht, 2009; Keshavan et al., 2010; Chen et al., 2021b).

Definition 1 (Incoherence). The incoherence parameters μ_1 and μ_2 of X^* are defined as:

$$\mu_1 := \frac{n_1}{r} \max_{1 \le i \le n_1} \left\| \boldsymbol{U}_{i,:}^{\star} \right\|_2^2 \quad and \quad \mu_2 := \frac{n_2}{r} \max_{1 \le j \le n_2} \left\| \boldsymbol{V}_{j,:}^{\star} \right\|_2^2.$$
(5)

It is self-evident that $1 \le \mu_1 \le n_1/r$ and $1 \le \mu_2 \le n_2/r$. In words, if the incoherence parameter μ_1 (resp. μ_2) is small, then the energy of U^* (resp. V^*) would be more or less dispersed across all rows of U^* (resp. V^*). Throughout this paper, for simplicity we denote

$$\mu = \max\{\mu_1, \mu_2\}$$
 and $n := \max\{n_1, n_2\}.$ (6)

• Turning to the zero-mean noise matrix E, we first introduce the following parameters:

$$\omega_{i,j}^{2} := \operatorname{Var}[E_{i,j}], \quad \omega_{\max}^{2} := \max_{i,j} \operatorname{Var}[E_{i,j}], \quad \omega_{\operatorname{row}}^{2} := \max_{i} \sum_{j=1}^{n_{2}} \operatorname{Var}[E_{i,j}], \quad \omega_{\operatorname{col}}^{2} := \max_{j} \sum_{i=1}^{n_{1}} \operatorname{Var}[E_{i,j}], \quad (7)$$

where $\omega_{i,j}, \omega_{\max}, \omega_{row}, \omega_{col} \ge 0$. Here, we allow the variances $\{\omega_{i,j}^2\}$ to be location-varying, in order to account for *heteroskedasticity* of noise. Moreover, we impose the following assumptions throughout:

Assumption 1 (Noise). Suppose the noise components satisfy the following properties:

- 1. The $E_{i,j}$'s are statistically independent and obey $\mathbb{E}[E_{i,j}] = 0$ for all $(i,j) \in [n_1] \times [n_2]$;
- 2. $\mathbb{P}(|E_{i,j}| > B) \leq n^{-12}$, where the quantity B satisfies

$$B \le C_{\mathsf{b}} \frac{\min\left\{\left(\omega_{\mathsf{row}}\omega_{\mathsf{col}}\right)^{1/2}, \omega_{\mathsf{row}}\right\}}{\sqrt{\log n}}$$

for some numerical constant $C_{\rm b} > 0$.

Remark 1. Assumption 1 imposes a mild condition on the tails of noise. For instance, if $\omega_{i,j} \simeq \omega_{\max}$ for all i, j, then B is allowed to be as large as $\min\{(n_1n_2)^{1/4}, \sqrt{n_2}\}\omega_{\max}$ (up to some logarithmic factor), which can be substantially larger than the typical noise level ω_{\max} . In comparisons to prior works, (i) this assumption is similar to - in fact slightly weaker than - Cai et al. (2021, Assumption 2) (in that the assumption therein requires noise distributions to be symmetric); (ii) given that Assumption 1 is satisfied if $\{E_{i,j}\}$ are $C\omega_{\max}$ -sub-Gaussian and $\omega_{\max} \lesssim \min\{(\omega_{row}\omega_{col})^{1/2}, \omega_{row}\}/\log n$, it is less stringent than the one assumed in Zhang et al. (2022, Theorem 4).

Goal. We seek to estimate the column subspace U^* (up to global rotation) on the basis of Y. Our goal is to design an estimator that satisfies the following two desirable properties simultaneously:

- 1) it allows for faithful estimation of the column subspace despite the presence of heteroskedasticity and unbalanced dimensionality; we hope to accomplish this for the widest possible range of SNRs;
- 2) it achieves the desirable statistical guarantees that do not degrade when the condition number $\kappa = \sigma_1^* / \sigma_r^*$ increases.

3 Algorithms

In this section, we proceed to describe the proposed algorithm in attempt to achieve the goal set forth in Section 2, following a brief overview of previous algorithms.

Review: SVD, diagonal-deleted PCA and HeteroPCA. Before continuing, we briefly review three popular methods that are commonly studied in the literature.

• The vanilla SVD-based approach. This approach computes the leading r singular vectors of Y, or equivalently, the top-r eigenspace of the Gram matrix YY^{\top} , namely,

$$(\text{vanilla SVD}) \qquad \widehat{U}_{\mathsf{svd}} \leftarrow \mathsf{eigs}_r(YY^{+}), \tag{8}$$

Algorithm 1: HeteroPCA(G_{in} , r, t_{max}) (Zhang et al., 2022)

- 1 input: symmetric matrix G_{in} , rank r, number of iterations t_{max} .
- 2 initialization: $G^0 = G_{in}$.
- 3
- for $t = 0, 1, ..., t_{\max}$ do $| U^t \Lambda^t U^{t\top} \leftarrow \text{rank-}r \text{ leading eigendecomposition of } G^t.$ 4
- $G^{t+1} = \mathcal{P}_{\mathsf{off}-\mathsf{diag}}\left(G^{t}\right) + \mathcal{P}_{\mathsf{diag}}\left(U^{t}\Lambda^{t}U^{t\top}\right).$ 5
- 6 output: matrix estimate $G = G^{t_{max}}$ and subspace estimate $U = U^{t_{max}}$.

where $eigs_r(\cdot)$ stands for the leading rank-r eigen-subspace of a matrix. While this approach works well when $n_2 = O(n_1)$, it suffers from some fundamental limitations in the case with $n_2 \gg n_1$ and heteroskedastic noise. To illustrate this point, direct calculation reveals that

$$\mathbb{E}\left[\boldsymbol{Y}\boldsymbol{Y}^{\top}\right] = \boldsymbol{X}^{\star}\boldsymbol{X}^{\star\top} + \operatorname{diag}\left(\left[\sum_{j=1}^{n_2} \mathbb{E}\left[E_{i,j}^2\right]\right]_{1 \le i \le n_1}\right).$$
(9)

When $n_2 \gg n_1$ and when the noise components are highly heteroskedastic, the set of diagonal entries $\left\{\sum_{j=1}^{n_2} \mathbb{E}\left[E_{i,j}^2\right]\right\}_{1 \le i \le n_1}$ might vary drastically, thereby resulting in a large deviation between the top-*r* eigenspace of $\mathbb{E}[\overline{Y}\overline{Y}^{\top}]$ and that of $X^*X^{*\top}$ (which is the desirable U^*).

• Diagonal-deleted PCA. In an effort to rectify the above limitation of the vanilla SVD-based approach, prior works have put forward a solution called "diagonal-deleted PCA," which suppresses the influence of the diagonal entries of YY^{\top} by suppressing them (Koltchinskii and Giné, 2000; Florescu and Perkins, 2016; Cai et al., 2021; Ndaoud et al., 2021; Ndaoud, 2022; Abbe et al., 2022); that is, this approach outputs

(diagonal-deleted PCA)
$$\widehat{U}_{\mathsf{del}} \leftarrow \mathsf{eigs}_r (YY^\top - \mathcal{P}_{\mathsf{diag}}(YY^\top)),$$
 (10)

where \mathcal{P}_{diag} denotes Euclidean projection onto the set of diagonal matrices. When the diagonal entries of $X^{\star}X^{\star \perp}$ are sufficiently small, we have

$$\mathbb{E}\left[\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Y}\boldsymbol{Y}^{\top}\right)\right] = \boldsymbol{X}^{\star}\boldsymbol{X}^{\star\top} - \mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{X}^{\star}\boldsymbol{X}^{\star\top}\right) \approx \boldsymbol{X}^{\star}\boldsymbol{X}^{\star\top} = \boldsymbol{U}^{\star}\boldsymbol{\Sigma}^{\star2}\boldsymbol{U}^{\star\top}$$

which forms the rationale of this approach.

• *The* HeteroPCA *algorithm*. The above diagonal-deleted approach can be further improved. Employing (10) as an initialization, Zhang et al. (2022) put forward the HeteroPCA algorithm that combines the spectral method with successively refined diagonal estimates; more precisely, HeteroPCA initializes Gas $\mathcal{P}_{off-diag}(YY^{\top})$, and alternates between the following two steps until convergence:

$$\begin{array}{ll} (\mathsf{HeteroPCA}) & \text{repeat} & (\mathrm{i}) & \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top \leftarrow \mathrm{rank}\text{-}r \text{ eigendecomposition of } (\boldsymbol{G}); \\ & (\mathrm{ii}) & \boldsymbol{G} \leftarrow \mathcal{P}_{\mathsf{off-diag}}\big(\boldsymbol{Y}\boldsymbol{Y}^\top\big) + \mathcal{P}_{\mathsf{diag}}\big(\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top\big). \end{array}$$

See Algorithm 1 for a complete description of this procedure, with the input matrix (or initialization) chosen to be $G_{in} = YY^{\top} - \mathcal{P}_{diag}(YY^{\top})$. The key lies in employing the improved diagonal estimates to help alleviate the bias induced by diagonal deletion.

When the condition number $\sigma_1^{\star}/\sigma_r^{\star}$ is large, however, the magnitude of the diagonal entries of $X^{\star}X^{\star\top}$ can be substantially larger than, say, the square of the least singular value of X^{\star} (i.e., $\sigma_r^{\star 2}$). If this is the case, then diagonal-deleted PCA might eraze a significant fraction of the useful signal, resulting in loss of effectiveness. This issue carries over to HeteroPCA, as its initialization — which is based on diagonal-deleted PCA might already be highly unreliable.

Algorithm 2: Deflated-HeteroPCA

1 input: data matrix \boldsymbol{Y} (cf. (3)), rank r, maximum number of iterations t_i , i = 1, 2, ...2 initialization: $k = 0, r_0 = 0, \boldsymbol{G}_0 = \mathcal{P}_{off-diag} (\boldsymbol{Y} \boldsymbol{Y}^\top)$. 3 while $r_k < r$ do 4 $\begin{vmatrix} k = k + 1. \\ \text{select } r_k \text{ via Eqn. (12).} \\ \boldsymbol{G}_k, \boldsymbol{U}_k \end{pmatrix} = \text{HeteroPCA}(\boldsymbol{G}_{k-1}, r_k, t_k)$. 7 output: subspace estimate $\boldsymbol{U} = \boldsymbol{U}_k$.

The proposed algorithm: Deflated-HeteroPCA. We now describe how to alleviate the above curse of ill-conditioning. One lesson that we have learned from past HeteroPCA theory (Zhang et al., 2022; Yan et al., 2021) is that: this procedure works well if (i) the condition number of the truth is well-controlled and (ii) the least singular value is not buried by noise. Motivated by this fact, we propose to divide the set of eigenvalues of interest into "well-conditioned" subblocks that are sufficiently separated from each other, and include more subblocks one by one. More precisely, the main ideas of the proposed algorithm are as follows:

- 1) Sequently identify a collection of ranks $r_0 = 0 < r_1 < r_2 < \cdots < r_{k_{\max}} = r$, which partitions the set of eigenvalues (or singular values) of interest into disjoint subblocks. These points are chosen to ensure that (i) $\sigma_{r_{k-1}+1}^{\star}/\sigma_{r_k}^{\star}$ is sufficiently small for each k, and (ii) there is a sufficient gap between $\sigma_{r_k}^{\star}$ and $\sigma_{r_k+1}^{\star}$. Given that we do not know the true signular values a priori, we shall make careful use of the singular values of our running estimates instead.
- 2) In the k-th round, we invoke HeteroPCA with the rank r_k and the initialization G_{k-1} to impute the diagonal entries and obtain an improved estimate G_k of the Gram matrix of interest. Here, the first iteration employs the diagonal-deleted version $G_0 = \mathcal{P}_{\text{off-diag}}(YY^{\top})$.

It then boils down to how to select the aforementioned ranks $\{r_k\}$ in a data-driven manner. Towards this end, we look at the following set of ranks in the k-th round:¹

$$\mathcal{R}_{k} := \left\{ r': r_{k-1} < r' \le r, \ \frac{\sigma_{r_{k-1}+1}\left(\boldsymbol{G}_{k-1}\right)}{\sigma_{r'}\left(\boldsymbol{G}_{k-1}\right)} \le 4 \text{ and } \sigma_{r'}\left(\boldsymbol{G}_{k-1}\right) - \sigma_{r'+1}\left(\boldsymbol{G}_{k-1}\right) \ge \frac{1}{r}\sigma_{r'}\left(\boldsymbol{G}_{k-1}\right) \right\}, \quad (11)$$

and select r_k as follows:

$$r_{k} = \begin{cases} \max \mathcal{R}_{k}, & \text{if } \mathcal{R}_{k} \neq \emptyset, \\ r, & \text{otherwise.} \end{cases}$$
(12)

Here, we remind the readers that $\sigma_i(\mathbf{G}_{k-1})$ is the *i*-th singular value of \mathbf{G}_{k-1} . Evidently, the first condition in (11) is imposed to ensure well-conditioning of each subblock, whereas the second condition in (11) aims to guarantee a sufficient spectral separation between adjacent subblocks.

In a nutshell, the proposed algorithm counters the bias effect initially incurred by diagonal deletion via successive "deflation", a term that we borrow from Dobriban and Owen (2019) (although the problem considered therein is drastically different). More concretely, we first estimate the first subblock (which contains the largest eigenvalues of interest) by means of the diagonal deletion idea; once we finish estimating the eigen-subspace associated with this subblock, we can readily compensate for the contribution of this subblock in the diagonal of interest. This strategy is then repeated subblock by subblock in order to successively reduce — or "deflate" — the original bias in the diagonal. For this reason, we refer to the proposed algorithm as Deflated-HeteroPCA, whose complete details are summarized in Algorithm 2.

4 Main theory

In this section, we demonstrate the desirable statistical performance for the proposed algorithm, which enjoys substantially improved dependency on the condition number. Before continuing, we find it helpful to

¹The threshold 4 in (11) can be replaced with any numerical constant $C_{gap} \geq 4$.

introduce the following rotation matrix for any $U \in \mathcal{O}^{n_1,r}$:

$$\boldsymbol{R}_{\boldsymbol{U}} = \operatorname*{arg\,min}_{\boldsymbol{R} \in \mathcal{O}^{r,r}} \left\| \boldsymbol{U} \boldsymbol{R} - \boldsymbol{U}^{\star} \right\|_{\mathrm{F}},\tag{13}$$

the one that best aligns U with U^* in the Euclidean sense; after all, it is in general infeasible to resolve the ambiguity brought by global rotation. As is well known in the literature (e.g., Ma et al. (2020, Section D.2.1)),

$$\boldsymbol{R}_{\boldsymbol{U}} = \operatorname{sgn}\left(\boldsymbol{U}^{\top}\boldsymbol{U}^{\star}\right),\tag{14}$$

where $sgn(\cdot)$ is defined in (2).

4.1 Spectral-norm-based statistical guarantees

Let us begin with statistical guarantees based on the spectral norm accuracy. The following theorem asserts that the proposed Deflated-HeteroPCA algorithm enjoys appealing theoretical guarantees in terms of the spectral norm error $||UR_U - U^*||$, no matter how large the condition number of Σ^* is. The proof of this theorem is deferred to Section A.

Theorem 1. Suppose that Assumption 1 holds. Assume that

$$\sigma_r^* \ge C_0 r \left(\omega_{\mathsf{col}} + \sqrt{\omega_{\mathsf{col}}\omega_{\mathsf{row}}}\right) \sqrt{\log n} \tag{15a}$$

$$\mu \le c_0 \frac{n_1}{r^3} \tag{15b}$$

$$0 < \mu r \omega_{\max}^2 \le \omega_{col}^2 \tag{15c}$$

for some sufficiently large (resp. small) constant $C_0 > 0$ (resp. $c_0 > 0$). If the numbers of iterations obey

$$t_k > \log\left(C\frac{\sigma_{r_{k-1}+1}^{\star 2}}{\sigma_{r_k+1}^{\star 2}}\right), \qquad 1 \le k < k_{\max}$$

$$(16a)$$

$$t_{k_{\max}} > \log\left(C\frac{\sigma_{r_{k_{\max}-1}+1}^{\star 2}}{\sigma_{\max}^2}\right) \tag{16b}$$

for some large enough constant C > 0, then with probability exceeding $1 - O(n^{-10})$, the output returned by Algorithm 2 satisfies

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \lesssim \frac{\omega_{\mathsf{col}}\sqrt{\log n}}{\sigma_{r}^{\star}} + \frac{\omega_{\mathsf{col}}\omega_{\mathsf{row}}\log n}{\sigma_{r}^{\star 2}}.$$
(17)

Here, $r_0 = 0, r_1, \ldots, r_{k_{\text{max}}}$ are the ranks selected in Algorithm 2 and k_{max} satisfies $r_{k_{\text{max}}} = r$.

We find it helpful to compare our theoretical guarantees with prior theory for this problem. To begin with, the prior theory Zhang et al. (2022) only covers the well-conditioned case; when κ is a bounded constant (as assumed therein), our statistical error bound (17) matches the one in Zhang et al. (2022, Theorem 4) (up to some logarithmic factors).² In addition, when it comes to the case where $\omega_{i,j} \simeq \omega_{\max}$ for all $(i, j) \in [n_1] \times [n_2]$, our error bound (17) simplifies to

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \lesssim rac{\sqrt{n_1 \log n} \, \omega_{\mathsf{max}}}{\sigma_r^{\star}} + rac{\sqrt{n_1 n_2 \log^2 n} \, \omega_{\mathsf{max}}^2}{\sigma_r^{\star 2}}$$

which matches the minimax lower bounds Cai et al. (2021, Theorem 2) and Cai and Zhang (2018, Theorem 4) (ignoring logarithmic factors). It is noteworthy that when $\omega_{i,j} \simeq \omega_{\max}$ for all $(i,j) \in [n_1] \times [n_2]$ and r = O(1), the signal-to-noise ratio condition (15a) simplifies to

$$\sigma_r^* \gtrsim \left[(n_1 n_2)^{1/4} + n_1^{1/2} \right] \omega_{\max} \sqrt{\log n} \tag{18}$$

which is necessary to ensure — up to logarithmic factor — the existence of a consistent estimator (which means the existence of an estimator \hat{U} obeying $\|\hat{U}R_{\hat{U}} - U^*\| = o(1)$) (see Cai et al. (2021, Theorem 2)).

²Zhang et al. (2022) establishes estimation guarantees for the sin Θ distance $\|\sin\Theta(\hat{U}, U^*)\|$, which is (nearly) equivalent to the metric $\min_{\boldsymbol{R}\in\mathcal{O}^{r\times r}} \|\hat{U}\boldsymbol{R} - U^*\|$ (or more precisely, $\|\sin\Theta(\hat{U}, U^*)\| \asymp \min_{\boldsymbol{R}\in\mathcal{O}^{r\times r}} \|\hat{U}\boldsymbol{R} - U^*\|$). See (Chen et al., 2021b, Lemma 2.6) for details.

4.2 Fine-grained $\ell_{2,\infty}$ -norm-based statistical guarantees

Moving beyond the spectral norm bounds, we proceed to the fine-grained $\ell_{2,\infty}$ -norm-based error bounds for column subspace estimation, which further capture how well the estimation error is spread out across the rows (Ma et al., 2020; Chen et al., 2020, 2019b, 2021c; Agterberg et al., 2022; Zhang and Zhou, 2022; Cai et al., 2022a). As has been shown in the literature, such $\ell_{2,\infty}$ -based subspace estimation guarantees play a crucial role in deriving performance bounds for the subsequent tasks like entrywise covariance estimation, entrywise tensor estimation, exact recovery in a variety of clustering and mixture models (Cai et al., 2021; Yan et al., 2021; Abbe et al., 2022; Cai et al., 2021; Abbe et al., 2022).

Before formally presenting our $\ell_{2,\infty}$ -norm-based result, we first introduce the following assumption on the noise matrix E.

Assumption 2. Suppose that the noise components satisfy Conditions 1 and 2 in Assumption 1. In addition, we assume that

$$\mathbb{P}\left(|E_{i,j}| > B\right) \le n^{-12},\tag{19}$$

where B satisfies, for some universal constant $C_{b} > 0$, that

$$B \le C_{\mathsf{b}}\omega_{\mathsf{max}} \frac{\min\left\{\left(n_1 n_2\right)^{1/4}, \sqrt{n_2}\right\}}{\log n}$$

Remark 2. With the logarithmic factors neglected, the only difference between Assumption 2 and Cai et al. (2021, Assumption 2) is that no symmetric distribution requirement is needed in Assumption 2.

Built upon Assumption 2, we derive the following $\ell_{2,\infty}$ -based theoretical guarantees for Deflated-HeteroPCA, with the proof postponed to Section B.

Theorem 2. Suppose that Assumption 2 holds and the signal-to-noise ratio satisfies

$$\frac{\sigma_r^{\star}}{\omega_{\max}} \ge C_0 r \left[(n_1 n_2)^{1/4} + n_1^{1/2} \right] \log n \tag{20a}$$

$$\mu \le c_0 \frac{n_1}{r^3} \tag{20b}$$

for some large (resp. small) enough constant $C_0 > 0$ (resp. $c_0 > 0$). If the numbers of iterations satisfy (16), then with probability exceeding $1 - O(n^{-10})$, then the estimate returned by Algorithm 2 satisfies

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}}-\boldsymbol{U}^{\star}\|_{2,\infty} \lesssim \sqrt{\frac{\mu r}{n_1}}\zeta_{\mathsf{op}},$$
 (21a)

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \lesssim \zeta_{\mathsf{op}},$$
 (21b)

where

$$\zeta_{\mathsf{op}} = \frac{\sqrt{n_1 n_2} \,\omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star 2}} + \frac{\sqrt{n_1} \,\omega_{\mathsf{max}} \log n}{\sigma_r^{\star}}.$$
(22)

Encouragingly, both the $\ell_{2,\infty}$ -based and spectral-norm-based estimation guarantees in (21) match the minimax lower bounds previously established in Cai et al. (2021, Theorem 2) (up to logarithmic factors), thus confirming the near minimax optimality of our results. It can also been seen from Cai et al. (2021, Theorem 2) that the signal-to-noise ratio requirement (20a) is, in general, essential (ignoring logarithmic factors) in order to enable the plausibility of consistent estimation.

Comparison with prior results. In order to demonstrate the utility of our algorithm and the accompanying theory, we compare our results with past works in the sequel. To ease presentation, the discussion below focuses attention on the case where $\mu, r = O(1)$.

• Requirement on the condition number κ . In order to obtain a consistent estimator³, all prior theory for both diagonal-deleted PCA (see Cai et al. (2021, Theorem 1)) and HeteroPCA (see Zhang et al. (2022, Theorem 4), Yan et al. (2021, Theorem 10) and Agterberg et al. (2022, Assumption 4)) assumes the condition number κ to obey

(prior requirement on
$$\kappa$$
) $\kappa \lesssim n_1^{1/4}$, (23)

in order to control the bias incurred during the diagonal deletion step. This, however, falls short of accommodating a wider range of condition numbers. In contrast, our result in Theorem 2 does not impose any assumptions on the condition number.

• Statistical error bounds. We now compare our statistical error bounds with the ones obtained in Cai et al. (2021); Agterberg et al. (2022); Yan et al. (2021). For notational convenience, define

$$\mathcal{E}_{\text{noise}} := \frac{\sqrt{n_1 n_2} \,\omega_{\max}^2 \log n}{\sigma_r^{\star 2}} + \frac{\kappa \omega_{\max} \sqrt{n_1 \log n}}{\sigma_r^{\star}},\tag{24}$$

which makes it more convenient for us to describe the previous results.

- Under the signal-to-noise ratio condition

$$\frac{\sigma_r^{\star}}{\omega_{\max}} \gtrsim \left(\kappa \left(n_1 n_2\right)^{1/4} + \kappa^3 n_1^{1/2}\right) \sqrt{\log n},\tag{25}$$

Cai et al. (2021, Theorem 1) asserts that the estimate \hat{U}_{del} returned by diagonal-deleted PCA obeys, with high probability,

$$\min_{\boldsymbol{R}\in\mathcal{O}^{r,r}} \left\| \widehat{\boldsymbol{U}}_{\mathsf{del}} \boldsymbol{R} - \boldsymbol{U}^* \right\|_{2,\infty} \lesssim \kappa^2 \sqrt{\frac{1}{n_1}} \big(\mathcal{E}_{\mathsf{noise}} + \mathcal{E}_{\mathsf{diag-del}} \big), \tag{26}$$

where $\mathcal{E}_{diag-del}$ is an additional error term due to the bias resulting from diagonal deletion.

- Focusing on the case where $n_2 \gtrsim n_1$, Agterberg et al. (2022, Theorem 2) establishes an $\ell_{2,\infty}$ error bound for the HeteroPCA estimate \hat{U}_{hpca} as follows:

$$\min_{\boldsymbol{R}\in\mathcal{O}^{r,r}} \left\| \widehat{\boldsymbol{U}}_{\mathsf{hpca}} \boldsymbol{R} - \boldsymbol{U}^{\star} \right\|_{2,\infty} \lesssim \sqrt{\frac{1}{n_1}} \mathcal{E}_{\mathsf{noise}},\tag{27}$$

albeit under a much more stringent SNR requirement:

$$\sigma_r^{\star} \gg \kappa \omega_{\max} \sqrt{n_2 \log n}. \tag{28}$$

- Yan et al. (2021, Theorem 10) further shows that under the same SNR condition (25), HeteroPCA yields an estimator \hat{U}_{hpca} with the following high-probability $\ell_{2,\infty}$ error bound:

$$\min_{\boldsymbol{R}\in\mathcal{O}^{r,r}}\left\|\widehat{\boldsymbol{U}}_{\mathsf{hpca}}\boldsymbol{R}-\boldsymbol{U}^{\star}\right\|_{2,\infty}\lesssim\kappa^{2}\sqrt{\frac{1}{n_{1}}}\mathcal{E}_{\mathsf{noise}}.$$
(29)

Let us compare our bounds with the above results. Recognizing that \mathcal{E}_{noise} is at least as large as ζ_{op} if we ignore logarithmic factors, our $\ell_{2,\infty}$ error bound (21a) improves the theoretical guarantees (26) and (29) by at least a factor of κ^2 . Additionally, our bound (21a) outperforms the bound (27) in terms of the dependency on κ (ignoring logarithmic factors).

• SNR requirement. Let us also briefly make comparisons regarding the SNR required for consistent estimation. To begin with, we make note that the vanilla SVD-based approach (cf. (8)) requires the SNR to exceed (Cai et al., 2021; Zhang et al., 2022)

$$\frac{\sigma_r^{\star}}{\omega_{\max}} \gtrsim \sqrt{n_1} + \sqrt{n_2},\tag{30}$$

³Here, a column subspace estimator \hat{U} is said to be consistent if $\min_{R \in \mathcal{O}^{r,r}} \|\hat{U}R - U^{\star}\| = o(1)$.

which can be substantially more stringent than the one required in (20a) if $n_2 \gg n_1$. In addition, compared with the SNR requirement imposed in the existing theory for diagonal-deleted PCA and HeteroPCA, our condition (20a) is weaker than the one used in Cai et al. (2021) and Yan et al. (2021) (see (25)) by at least a factor of κ , while at the same time being weaker than the condition (28) assumed in Agterberg et al. (2022) by a factor of $\kappa(n_2/n_1)^{1/4}$ when $n_2 \gg n_1$.

High-level proof strategy. While the proofs of our main theorems are deferred to the Appendix, we highlight some novelty and technical challenges in our proof. In an attempt to obtain fine-grained $\ell_{2,\infty}$ control while remaining condition-number-free, we develop a new proof strategy that differs drastically from the state-of-the-art techniques based on leave-one-out decoupling arguments (Yan et al., 2021; Cai et al., 2021). Inspired by a spectral representation lemma derived in the recent work Xia (2021) (see also Lemma 1), we proceed by decomposing the difference between the subspaces into an infinite sum of polynomials of the error matrix. With this decomposition at hand, one major part of our proof hinges upon establishing sharp $\ell_{2,\infty}$ bounds on each of the polynomials of the error matrix. The key challenge for this part lies in how to deal with the complicated and accumulated dependence brought by the power of the error matrix, for which we resort to careful induction analyses. We will then single out several sequences of critical quantities and develop intricate arguments to control these quantities in a recursive and inductive manner.

5 Consequences for specific models

To better illustrate the effectiveness of the proposed algorithm, we develop concrete consequences of our theory in Section 4 for two specific models. In each case, we shall begin by describing the model, followed by concrete algorithms and theory tailored to the specific model.

5.1 Factor models and spiked covariance models

Model. A frequently studied model employed to capture low-dimensional structure in high-dimensional sample data is the factor model, which finds applications numerous contexts including finance and econometrics (Lawley and Maxwell, 1962; Fan et al., 2020, 2021), functional magnetic resonance imaging (Chen et al., 2015), and signal processing (Zhao et al., 1986; Kritchman and Nadler, 2008, 2009), to name just a few. For concreteness, suppose that we observe a collection of n independent sample vectors in \mathbb{R}^d generated as follows:

$$\boldsymbol{y}_j = \boldsymbol{B}^{\star} \boldsymbol{f}_j + \boldsymbol{\varepsilon}_j \in \mathbb{R}^d, \tag{31a}$$

where $B^{\star} \in \mathbb{R}^{d \times r}$ represents the factor loading matrix with $r \ll d$, $\{f_j\}$ stands for the latent factor vectors, and $\{\varepsilon_j\}$ denotes the noise vectors. We assume that

$$\boldsymbol{B}^{\star} = \boldsymbol{U}^{\star} \boldsymbol{\Lambda}^{\star 1/2} \in \mathbb{R}^{d \times r} \quad \text{and} \quad \boldsymbol{f}_{j} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{I}_{r}\right), \quad 1 \leq j \leq n,$$
(31b)

with $U^* \in \mathcal{O}^{d,r}$ and $\Lambda^* = \text{diag}(\lambda_1^*, \ldots, \lambda_r^*)$ being a diagonal matrix containing all eigenvalues of $B^* B^{*\top}$. Equivalently, one can express it as the following spiked covariance model:

$$\boldsymbol{y}_j = \boldsymbol{x}_j + \boldsymbol{\varepsilon}_j, \quad \text{with } \boldsymbol{x}_j \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{U}^* \boldsymbol{\Lambda}^* \boldsymbol{U}^{*\top}\right), \quad 1 \leq j \leq n.$$
 (32)

The noise vectors are allowed to be heteroskedastic, and it is assumed that

• the $\varepsilon_{i,j}\mbox{'s}$ are statistically independent, zero-mean, and $\omega\mbox{-sub-Gaussian},$

where $\omega > 0$ is an upper bound on the sub-Gaussian norm of any noise entry. We also assume that

$$\|\boldsymbol{U}^{\star}\|_{2,\infty} \le \sqrt{\frac{\mu_{\mathsf{pc}}r}{d}}.$$
(33)

Our goal is to estimate the subspace U^* based on the observed vectors $\{y_i\}_{1 \le i \le n}$.

Algorithm and theoretical guarantees. Taking the data matrix as $Y = [y_1 \dots y_n] \in \mathbb{R}^{d \times n}$, we can readily invoke Algorithm 2 to estimate the subsapce U^* . The performance guarantees are stated below, whose proof is deferred to Section C.1.

Corollary 1. Consider the factor model in (31). Assume that

$$\frac{\lambda_r^*}{\omega^2} \ge C_1 r^2 \left[\left(\frac{d}{n}\right)^{1/2} + \frac{d}{n} \right] \log^2(n+d), \tag{34a}$$

$$\mu_{\mathsf{pc}} \vee \log(n+d) \le c_1 \frac{d}{r^3},\tag{34b}$$

$$r \vee \log(n+d) \le c_1 n \tag{34c}$$

for some sufficiently large (resp. small) constant $C_1 > 0$ (resp. $c_1 > 0$). Suppose that the numbers of iterations obey, for some large enough constant C > 0,

$$t_k \ge \log_2\left(C\frac{\lambda_{r_{k-1}+1}^*}{\lambda_{r_k+1}^*}\right), \quad \forall 1 \le k \le k_{\max} - 1,$$
(35a)

$$t_{k_{\max}} \ge \log\left(C\frac{n\lambda_{r_{k_{\max}-1}+1}^{\star}}{\omega^2}\right),\tag{35b}$$

where k_{\max} satisfies $r_{k_{\max}} = r$. Then with probability exceeding $1 - O((n+d)^{-10})$, the output U returned by Algorithm 2 satisfies

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\|_{2,\infty} \lesssim \sqrt{\frac{(\mu_{\mathsf{pc}} + \log(n+d))\,r}{d}} \left(\frac{\sqrt{d/n}\,\omega^2\log^2(n+d)}{\lambda_r^{\star}} + \frac{\sqrt{d/n}\,\omega\log(n+d)}{\sqrt{\lambda_r^{\star}}}\right),\tag{36a}$$

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \lesssim \frac{\sqrt{d/n}\,\omega^2\log^2(n+d)}{\lambda_r^{\star}} + \frac{\sqrt{d/n}\,\omega\log(n+d)}{\sqrt{\lambda_r^{\star}}}.$$
(36b)

Let us briefly discuss the implications of our results. Consider, for example, the case where $\mathbb{E}[\varepsilon_{i,j}^2] \simeq \sigma^2$ for all $(i, j) \in [d] \times [n]$. The spectral norm bound (36b) matches the minimax limit (see Zhang et al. (2022, Theorems 1 and 4)) modulo some logarithmic factor. In addition, recognizing that

$$d \left\| \boldsymbol{U} \boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star} \right\|_{2,\infty}^{2} \geq \left\| \boldsymbol{U} \boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star} \right\|_{\mathrm{F}}^{2} \geq \left\| \boldsymbol{U} \boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star} \right\|^{2},$$

we see that the $\ell_{2,\infty}$ bound (36a) is also near-optimal when $\mu_{pc}, r \approx 1$. Again, our result does not rely on the condition number $\kappa_{pc} = \lambda_1^* / \lambda_r^*$. Moreover, Zhang et al. (2022, Theorem 1) assumes that κ_{pc} is bounded by a numerical constant, while (Cai et al., 2021, Corollary 2) requires $\kappa_{pc} \lesssim \sqrt{\frac{d}{\mu r}}$; these form another aspect in which Corollary 1 improves upon the prior literature.

5.2 Tensor PCA

Model. Another canonical example in which column subspace estimation plays a key role is tensor PCA (or low-rank tensor estimation), a problem that has been studied extensively in recent literature (Richard and Montanari, 2014; Zhang and Xia, 2018; Cai et al., 2021, 2022a; Han et al., 2022b; Zhou et al., 2022; Han and Zhang, 2022). To be presice, assume that we observe a noisy tensor as follows:

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}}^{\star} + \boldsymbol{\mathcal{E}} \in \mathbb{R}^{n_1 \times n_2 \times n_3},\tag{37a}$$

where \mathcal{X}^{\star} is an unknown low-rank tensor to be estimated, and \mathcal{E} represents the noise tensor. We assume that \mathcal{X}^{\star} has low-Tucker-rank in the sense that (Zhang et al., 2022; Han and Zhang, 2022; Xia et al., 2022)

$$\boldsymbol{\mathcal{X}}^{\star} = \boldsymbol{\mathcal{S}}^{\star} \times_{1} \boldsymbol{U}_{1}^{\star} \times_{2} \boldsymbol{U}_{2}^{\star} \times_{3} \boldsymbol{U}_{3}^{\star}, \tag{37b}$$

where the core tensor S^* lies in $\mathbb{R}^{r_1 \times r_2 \times r_3}$ (with small r_1, r_2, r_3), and the tensor "principal components" $U_i^* \in \mathcal{O}^{n_i, r_i}$ $(1 \le i \le 3)$ satisfy the incoherence condition

$$\|\boldsymbol{U}_{i}^{\star}\|_{2,\infty} \leq \sqrt{\frac{\mu r_{i}}{n_{i}}}, \qquad 1 \leq i \leq 3.$$

$$(38)$$

Moreover, the noise tensor $\mathcal{E} = [E_{i,j,k}]_{(i,j,k)\in[n_1]\times[n_2]\times[n_3]}$ is composed of independent entries such that

• the $E_{i,j,k}$'s are statistically independent, zero-mean, and ω -sub-Gaussian,

where $\omega > 0$ is an upper bound on the sub-Gaussian norm of each noise entry. The aim is to compute a faithful estimate of the true tensor \mathcal{X}^* as well as the principal components U_1^*, U_2^* and U_3^* .

Additional notation. Before presenting the algorithm and our theoretical results, we introduce several useful notation. For any $1 \le i \le 3$ and $1 \le j \le r_i$, we denote by $\sigma_{i,j}^*$ the *j*-th largest singular value of the *i*-th matricization of \mathcal{X} — denoted by $\mathcal{M}_i(\mathcal{X})$. Define

$$\sigma_{\min}^{\star} := \min\left\{\sigma_{1,r_{1}}^{\star}, \sigma_{2,r_{2}}^{\star}, \sigma_{3,r_{3}}^{\star}\right\}$$

and the condition number of the true tensor is then defined as

$$\kappa := \frac{\max\left\{\sigma_{1,1}^{\star}, \sigma_{2,1}^{\star}, \sigma_{3,1}^{\star}\right\}}{\sigma_{\min}^{\star}}.$$

For any $1 \leq i \leq 3$, we also let $r_{i,1}, r_{i,2}, \ldots, r_{i,k_{\max}^i}$ denote the ranks selected in Algorithm 2 if we apply this algorithm with the input matrix $\mathbf{Y} = \mathcal{M}_i(\mathbf{Y})$, the rank r_i , and the numbers of iterations $t_{i,1}, \ldots, t_{i,k_{\max}^i}$. As usual, we choose k_{\max}^i such that $r_{k_{\max}^i} = r_i$. In addition, for notational convenience we let

$$n = \max_{1 \le i \le 3} n_i$$
 and $r = \max_{1 \le i \le 3} r_i$,

and define

$$oldsymbol{U}_4^\star = oldsymbol{U}_1^\star$$
 and $oldsymbol{U}_5^\star = oldsymbol{U}_2^\star$

Algorithm and statistical guarantees. In order to apply Deflated-HeteroPCA, let us look at the matrix $\mathcal{M}_i(\mathcal{X}^*) \in \mathbb{R}^{n_i \times (n_1 n_2 n_3)/n_i}$, the *i*-th matricization of \mathcal{X}^* . Recognizing that U_i^* is also the left singular space of $\mathcal{M}_i(\mathcal{X}^*)$ since

$$\mathcal{M}_{i}\left(oldsymbol{\mathcal{X}}^{\star}
ight) = oldsymbol{U}_{i}^{\star} \mathcal{M}_{i}\left(oldsymbol{\mathcal{S}}^{\star}
ight) \left(oldsymbol{U}_{i+2}^{\star} \otimes oldsymbol{U}_{i+1}^{\star}
ight),$$

we propose to apply the Deflated-HeteroPCA algorithm to compute an initial subspace estimate \hat{U}_i^0 for U_i^{\star} . Armed with these initial estimates, we invoke the high-order orthogonal iteration (HOOI) algorithm (De Lathauwer et al., 2000b; Zhang and Xia, 2018) to iteratively refine the estimates. More specifically, in the *t*-th iteration, we calculate

 $\widehat{U}_{i}^{t} = \text{ the first } r \text{ left singular vectors of } \mathcal{M}_{i} \big(\mathcal{Y} \times_{i+1} \widehat{U}_{i+1}^{t-1} \times_{i+2} \widehat{U}_{i+2}^{t-1} \big), \quad 1 \leq i \leq 3,$

where i + 1 and i + 2 are calculated modulo 3. Once the above iterative procedure converges, we employ the resulting subspace estimates $\hat{U}_1, \hat{U}_2, \hat{U}_3$ to construct the following estimator for the true tensor:

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{Y}} imes_1 \mathcal{P}_{\widehat{\boldsymbol{U}}_1} imes_2 \mathcal{P}_{\widehat{\boldsymbol{U}}_2} imes_3 \mathcal{P}_{\widehat{\boldsymbol{U}}_3},$$

where we recall the notation $\mathcal{P}_{U} = UU^{\top}$.

The whole procedure is summarized in Algorithm 3, where Deflated-HeteroPCA($\mathbf{Y}, r, t_1, \ldots, t_{max}$) is the output of Algorithm 2 with the input matrix \mathbf{Y} , the rank r, and the numbers of iterations t_1, \ldots, t_{max} . Our main theory for Deflated-HeteroPCA readily leads to the following statistical guarantees for Algorithm 3.

Algorithm 3: High-order orthogonal iteration (HOOI) (De Lathauwer et al., 2000b; Zhang and Xia, 2018)

1 input: \mathcal{Y} , ranks r_1, r_2, r_3 , number of iterations $\{t_{i,j}\}_{1 \le i \le 3, 1 \le j \le k_{\max}^i}$ and t_{\max} .

2 initialization: call Algorithm 2 to compute

$$\begin{split} \widehat{\boldsymbol{U}}_1^0 &= \mathsf{Deflated}\text{-}\mathsf{HeteroPCA}\left(\mathcal{M}_1(\boldsymbol{\mathcal{Y}}), r_1, t_{1,1}, t_{1,2}, \dots, t_{1,k_{\mathsf{max}}^1}\right);\\ \widehat{\boldsymbol{U}}_2^0 &= \mathsf{Deflated}\text{-}\mathsf{HeteroPCA}\left(\mathcal{M}_2(\boldsymbol{\mathcal{Y}}), r_2, t_{2,1}, t_{2,2}, \dots, t_{2,k_{\mathsf{max}}^2}\right);\\ \widehat{\boldsymbol{U}}_3^0 &= \mathsf{Deflated}\text{-}\mathsf{HeteroPCA}\left(\mathcal{M}_3(\boldsymbol{\mathcal{Y}}), r_3, t_{3,1}, t_{3,2}, \dots, t_{3,k_{\mathsf{max}}^3}\right). \end{split}$$

while $t < t_{max} do$

 $\begin{array}{l} \mathbf{w} \text{ infect} < t_{\text{max}} \text{ d} \mathbf{\mathcal{G}} \\ \mathbf{3} \\ \hat{U}_1^t = \text{ leading } r_1 \text{ left singular vectors of } \mathcal{M}_1(\boldsymbol{\mathcal{Y}} \times_2 \widehat{U}_2^{t-1} \times_3 \widehat{U}_3^{t-1}). \\ \hat{U}_2^t = \text{ leading } r_2 \text{ left singular vectors of } \mathcal{M}_1(\boldsymbol{\mathcal{Y}} \times_3 \widehat{U}_3^{t-1} \times_1 \widehat{U}_1^{t-1}). \\ \mathbf{5} \\ \hat{U}_3^t = \text{ leading } r_3 \text{ left singular vectors of } \mathcal{M}_3(\boldsymbol{\mathcal{Y}} \times_1 \widehat{U}_1^{t-1} \times_2 \widehat{U}_2^{t-1}). \end{array}$ 6 compute $\hat{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{Y}} \times_1 \hat{U}_1^{t_{\text{max}}} \hat{U}_1^{t_{\text{max}}} \times_2 \hat{U}_2^{t_{\text{max}}} \hat{U}_2^{t_{\text{max}}} \times_3 \hat{U}_3^{t_{\text{max}}} \hat{U}_3^{t_{\text{max}}}^{t_{\text{max}}}.$ 7 output: subspace estimates $\hat{U}_1 = \hat{U}_1^{t_{\text{max}}}, \ \hat{U}_2 = \hat{U}_2^{t_{\text{max}}}, \ \hat{U}_3 = \hat{U}_3^{t_{\text{max}}}, \text{ and tensor estimate } \hat{\boldsymbol{\mathcal{X}}}.$

Corollary 2. Consider the tensor PCA model in (37). Suppose that $n_1 \approx n_2 \approx n_3 \approx n$, and

$$\frac{\sigma_{\min}^{\star}}{\omega} \ge C_2 r n^{3/4} \log n \tag{39a}$$

$$\mu \le c_2 \sqrt{\frac{n}{r^4}} \tag{39b}$$

for some sufficiently large (resp. small) constant $C_2 > 0$ (resp. $c_2 > 0$). For any $1 \le i \le 3$, if one chooses

$$t_{i,1} \ge \log_2\left(C\frac{\sigma_{i,r_{i,k-1}+1}^{\star 2}}{\sigma_{i,r_{i,k}+1}^{\star 2}}\right), \quad 1 \le k \le k_{\max}^i - 1,$$
(40a)

$$t_{i,k_{\max}^{i}} \ge \log\left(C\frac{\sigma_{r_{i,k_{\max}-1}}^{\star 2}+1}{\omega^{2}}\right),\tag{40b}$$

then with probability exceeding $1 - O(n^{-10})$, the initial estimator \widehat{U}_i^0 satisfies

$$\left\|\widehat{U}_{i}^{0}\boldsymbol{R}_{\widehat{U}_{i}^{0}}-\boldsymbol{U}_{i}^{\star}\right\|_{2,\infty} \lesssim \frac{\mu r}{\sqrt{n}} \left(\frac{n^{3/2}\omega^{2}\log^{2}n}{\sigma_{\min}^{\star^{2}}}+\frac{\sqrt{n}\,\omega\log n}{\sigma_{\min}^{\star}}\right),\tag{41a}$$

$$\left\|\widehat{\boldsymbol{U}}_{i}^{0}\boldsymbol{R}_{\widehat{\boldsymbol{U}}_{i}^{0}}-\boldsymbol{U}^{\star}\right\| \lesssim \frac{n^{3/2}\omega^{2}\log^{2}n}{\sigma_{\min}^{\star2}} + \frac{\sqrt{n}\,\omega\log n}{\sigma_{\min}^{\star2}}.$$
(41b)

In addition, if the number of iterations in HOOI obeys $t_{\max} \ge C(\log(\frac{n}{\sigma_{\min}}) \lor 1)$ for some large enough constant C > 0, then with probability exceeding $1 - O(n^{-10})$ one has

$$\left\|\widehat{\boldsymbol{U}}_{i}\boldsymbol{R}_{\widehat{\boldsymbol{U}}_{i}}-\boldsymbol{U}_{i}^{\star}\right\| \lesssim \frac{\sqrt{n_{i}}\,\omega}{\sigma_{\min}^{\star}}, \qquad 1 \le i \le 3$$
(42a)

$$\left\|\widehat{\boldsymbol{\mathcal{X}}} - \boldsymbol{\mathcal{X}}^{\star}\right\|_{\mathrm{F}}^{2} \lesssim \left(n_{1}r_{1} + n_{2}r_{2} + n_{3}r_{3}\right)\omega^{2}.$$
(42b)

The bounds in (42) are rate-optimal, since they match the minimax lower bounds established for the i.i.d. Gaussian noise case in Zhang and Xia (2018, Theorem 3). This confirms that the proposed Deflated-HeteroPCA algorithm serves as an effective paradigm to initialize the HOOI algorithm. It is also noteworthy that when r = O(1), the SNR condition (39) is essential (ignoring logarithmic factor) to ensure that consistent estimation is computable within polynomial time; see Zhang and Xia (2018, Theorem 4).



Figure 2: Estimation errors of U for Deflated-HeteroPCA, Diagonal-deleted PCA, HeteroPCA and Vanilla SVD for r = 3. Plot (a) (resp. (b)) reports the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the noise level ω (where $n_1 = 100, n_2 = 1,000, \kappa = 5$). Plot (c) (resp. (d)) shows the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the column dimension κ (where $n_1 = 100, n_2 = 1,000, \omega = 1$). Plot (e) (resp. (f)) displays the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the condition number n_2 (where $n_1 = 100, \kappa = 5, \omega = 1$).

It is then helpful to compare our results with the prior works Zhang and Xia (2018) and Han et al. (2022b). Firstly, Zhang and Xia (2018, Theorem 1) assumes that the noise tensor has i.i.d. Gaussian entries, which is clearly much more stringent than our result. Secondly, while Han et al. (2022b, Theorem 4.1) allows the noise to be heteroskedastic, it requires the condition number of the tensor to be bounded (see the analysis for their main theorems); in comparison, our theory in Corollary 2 suggests that Algorithm 3 succeeds no matter how large the condition number κ is.

6 Numerical experiments

In this section, we conduct additional numerical experiments to verify the practical applicability of our algorithm. All results in this section are averaged over 50 Monte Carlo runs.

Low-rank subspace estimation from noisy observation. To begin with, we consider the problem of estimating the column subspace of X^* from the noisy data (3). We randomly generate $U^* \in \mathcal{O}^{n_1,r}$ and $V^* \in \mathcal{O}^{n_2,r}$, and $X^* = U^* \Sigma^* V^{*\top}$, where $\Sigma^* = \text{diag}(\sigma_1^*, \ldots, \sigma_r^*)$. For each $i \in [n_1]$, we independently and uniformly draw $\omega_i \in [0, \omega]$, whereas the $E_{i,j}$'s are independently drawn from $\mathcal{N}(0, \omega_i^2)$. We fix $n_1 = 100$, set $\sigma_r^* = (n_1 n_2)^{1/4} + n_1^{1/2}$, and consider the following two settings: (i) r = 3, $\sigma_1^* = \kappa \sigma_3^*$ and $\sigma_2^* = \sigma_3^*$; (ii) r = 5, $\sigma_1^* = \kappa \sigma_5^*$, $\sigma_2^* = \sigma_3^* = \kappa^{1/2} \sigma_5^*$ and $\sigma_4^* = \sigma_5^*$. We report the spectral-norm-based error $\|UR_U - U^*\|$ and the $\ell_{2,\infty}$ error $\|UR_U - U^*\|_{2,\infty}$ for each of the following four algorithms: (a) Deflated-HeteroPCA in Algorithm 2, where the numbers of iterations are chosen to be $t_i = 10$; (b) the diagonal-deleted PCA procedure as in (10); (c) HeteroPCA in Algorithm 1, where the number of iterations is taken to be 100; (d) the vanilla SVD-based approach described in (8). The results for r = 3 and r = 5 are reported in Figures 2 and 3, respectively. As



Figure 3: Estimation errors of U for Deflated-HeteroPCA, Diagonal-deleted PCA, HeteroPCA and Vanilla SVD when r = 5. Plot (a) (resp. (b)) displays the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the noise level ω (where $n_1 = 100, n_2 = 1,000, \kappa = 5$). Plot (c) (resp. (d)) shows the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the condition number κ (where $n_1 = 100, n_2 = 1,000, \omega = 1$). Plot (e) (resp. (f)) diaplays the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the condition number dimension n_2 (where $n_1 = 100, \kappa = 5, \omega = 1$).

can be seen from the plots, the proposed Deflated-HeteroPCA algorithm significantly outperforms the other three methods, and it is the only algorithm whose performance is unaffected by the condition number κ .

Factor model. We then turn attention to the factor model (32). We consider the case with d = 100, r = 3, and randomly generate the subspace $U^* \in \mathcal{O}^{d,3}$ and $F = [f_1 \dots f_n] \in \mathbb{R}^{3 \times n}$ with i.i.d. standard Gaussian entries. We set the diagonal matrix $\Lambda^* = \text{diag}(\lambda_1^*, \lambda_2^*, \lambda_3^*)$ with $\lambda_1^* = \kappa \lambda_3^*$ and $\lambda_2^* = \lambda_3^* = (d/n)^{1/2} + d/n$. The noise matrix is generated in the same way as in the previous setting. We report in Figure 4 the ℓ_2 and $\ell_{2,\infty}$ errors for the principal subspace for the four methods, Deflated-HeteroPCA, Diagonal-deleted PCA, HeteroPCA and Vanilla SVD. The numerical results suggest that the proposed Deflated-HeteroPCA algorithm achieves the best performance among all these methods, which is not affected as κ_{pc} varies.

Tensor PCA. Finally, we conduct numerical experiments for the tensor PCA model (37). We fix n = 50and r = 3, and introduce a quantity $\sigma^* = n^{3/4}$. The subspaces $U_1^* \in \mathcal{O}^{100,3}$, $U_2^* \in \mathcal{O}^{100,3}$ and $U_3^* \in \mathcal{O}^{100,3}$ are generated randomly, and the core tensor $\mathcal{S}^* \in \mathbb{R}^{3\times3}$ is a diagonal tensor with entries $S_{1,1,1} = \kappa \sigma^*$ and $S_{2,2,2} = S_{3,3,3} = \sigma^*$. The noise tensor is generated in the following way: we first generate three random vectors α, β and γ , where $\{\alpha_i\}, \{\beta_j\}, \{\gamma_k\}$ are independently drawn from [0,1]. We then generate each $E_{i,j,k}$ independently from $\mathcal{N}(0, \omega^2 \alpha_i^2 \beta_j^2 \gamma_k^2)$. The above four subspace estimation methods are applied to obtain initial subspace estimates, followed by 50 iterations of HOOI to refine the subspace estimators and construct the final tensor estimates. Figures 5 and 6 report the initial subspace estimation errors and the final subspace/tensor estimation errors, respectively. We can see from these plots that the Deflated-HeteroPCA algorithm produces faithful initial estimators in terms of both the ℓ_2 and $\ell_{2,\infty}$ errors, outperforming the other three methods. Moreover, compared with the other three methods, the Deflated-HeteroPCA algorithm serves as a more effective initialization scheme that can help one achieve more reliable subspace and tensor



Figure 4: Estimation errors of U for Deflated-HeteroPCA, Diagonal-deleted PCA, HeteroPCA and Vanilla SVD under the factor model (32) when r = 3. Plot (a) (resp. (b)) displays the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the noise level ω (where $d = 100, n = 1,000, \kappa_{pc} = 100$). Plot (c) (resp. (d)) shows the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the condition number κ_{pc} (where $d = 100, n = 1,000, \omega = 1$). Plot (e) (resp. (f)) displays the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the sample size n (where $d = 100, \kappa_{pc} = 100, \omega = 1$).

estimators.

7 Related works

This paper is closely related to the problem of matrix denoising, which aims to estimate either a low-rank matrix or its column subspace based on noisy observations and spans a diverse array of applications (Chen et al., 2021b). In addition to the examples of factor models and tensor estimation (Cai and Zhang, 2018; Cai et al., 2021; Zhu et al., 2019; Richard and Montanari, 2014; Zhang and Xia, 2018; Cai et al., 2021), it can also help us understand and solve several clustering problems (Rohe et al., 2011; Florescu and Perkins, 2016; Cai et al., 2021; Chen et al., 2022; Cai and Zhang, 2018; Löffler et al., 2021; Ndaoud, 2022; Srivastava et al., 2022; Han et al., 2022a; Zhang and Zhou, 2022). When it comes to the task of estimating the whole matrix, a number of methods have been put forward and thoroughly studied in the literature, including but not limited to singular value hard thresholding (Gavish and Donoho, 2014; Chatterjee, 2015), singular value soft thresholding (Cai et al., 2010; Koltchinskii et al., 2011; Donoho and Gavish, 2014) and singular value shrinkage (Nadakuditi, 2014; Gavish and Donoho, 2017). Turning to the task of subspace estimation, the vanilla SVD-based approach (see (8)) has been commonly used and widely studied in the literature (Koltchinskii and Xia, 2016; Cai and Zhang, 2018; Bao et al., 2021; Xia, 2021; Chen et al., 2021b). How to perform uncertainty quantification for this approach has also been demonstrated in the previous work (see (Chen et al., 2021b)). In the scenario where the matrix dimensions are extremely unbalanced and the noise is heteroskedastic, however, such estimators can be highly suboptimal for subspace estimation. As already mentioned previously, the diagonal-deleted PCA and HeteroPCA algorithms have been proposed to improve the performance over the vanilla SVD approach (Cai et al., 2021; Zhang et al., 2022; Agterberg et al., 2022; Yan et al., 2021). In fact, it has also been shown in Yan et al. (2021) that the HeteroPCA admits a



Figure 5: Initial estimation errors of \widehat{U}_1^0 for Deflated-HeteroPCA, Diagonal-deleted PCA, HeteroPCA and Vanilla SVD under the tensor SVD model (32). Plot (a) (resp. (b)) diplays the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the noise level ω (where $n_1 = n_2 = n_3 = 50, r = 3, \kappa = 6$). Plot (c) (resp. (d)) shows the ℓ_2 (resp. $\ell_{2,\infty}$) error vs. the condition number κ (where $n_1 = n_2 = n_3 = 50, r = 3, \kappa = 6$). Plot (c) (resp. (d)) shows the ℓ_2 (resp. $\ell_{2,\infty}$) error vs.

non-asymptotic distributional theory, which paves the way to construction of fine-grained confidence regions for this problem. Another family of effective algorithms — which can even accommodate the case when there is additional prior structure on the low-rank factors — is approximate message passing (Montanari and Venkataramanan, 2021; Deshpande et al., 2017; Feng et al., 2022; Li et al., 2023; Li and Wei, 2022; Montanari and Wu, 2022), for which the existing theory often requires more stringent assumptions on the noise components (e.g., i.i.d. Gaussian). It is also worth mentioning that how to accelerate optimization-based low-rank estimation algorithms in spite of ill conditioning has been an active research topic as well, which oftentimes involves proper preconditioning (Tong et al., 2021; Xu et al., 2023); the statistical guarantees therein, however, are still dependent on the condition number.

With regards to the factor model, one can easily find numerous works on this topic. The model (32) has been extensively studied under the names of spiked covariance models (Johnstone, 2001; Paul, 2007; Bai and Ding, 2012; Wang and Fan, 2017; Donoho et al., 2018; Perry et al., 2018; Bao et al., 2022) and factor models (Lawley and Maxwell, 1962; Bai and Li, 2012; Fan et al., 2016; Bai and Wang, 2016). Focusing on principal component estimation under heteroskedastic noise, Hong et al. (2016, 2018a,b) investigate the case where the noise components within each noise vector ε_j are i.i.d., and develop asymptotic analysis for PCA and a variant called Weighted PCA. Turning to non-asymptotic analysis, the theoretical performances of diagonal-deleted PCA (Cai et al., 2021) and HeteroPCA have been investigated in (Cai et al., 2021; Zhang et al., 2022; Yan et al., 2021). It is also worth noting that principal component estimation in the presence of missing data encounters additional challenges (Cai et al., 2021; Zhang et al., 2022; Zhu et al., 2019; Pavez and Ortega, 2020; Yan et al., 2021), which is beyond the scope of this work.

Another important example considered in this paper is the tensor PCA or tensor SVD model (37). Under this model, Richard and Montanari (2014); Hopkins et al. (2015); Anandkumar et al. (2017); Arous et al. (2019); Perry et al. (2020) study the statistical and computational limits for rank-1 tensors. For low Tucker-rank tensors, many methods have been proposed for tensor/subspace estimation, including high-



Figure 6: Final estimation errors of \hat{U}_1 and $\hat{\mathcal{X}}$ for Deflated-HeteroPCA, Diagonal-deleted PCA, HeteroPCA and Vanilla SVD under the tensor SVD model (32). We report (a) (resp. (b)) ℓ_2 (resp. Frobenious) error of \hat{U}_1 (resp. $\hat{\mathcal{X}}$) vs. noise level ω (where $n_1 = n_2 = n_3 = 50, r = 3, \kappa = 6$); (c) (resp. (d)) ℓ_2 (resp. Frobenious) error of \hat{U}_1 (resp. $\hat{\mathcal{X}}$) vs. condition number κ (where $n_1 = n_2 = n_3 = 50, r = 3, \omega = 2$).

order SVD (HOSVD, De Lathauwer et al. (2000a)), high-order orthogonal iteration (HOOI, De Lathauwer et al. (2000b); Zhang and Xia (2018)), the sequentially truncated higher-order singular value decomposition algorithm (ST-HOSVD, Vannieuwenhoven et al. (2012)), projected gradient descent (Han et al., 2022b), and scaled gradient descent (Tong et al., 2022). When the noise tensor has i.i.d. Gaussian entries, Zhang and Xia (2018) proves the statistical and computational limit for the tensor SVD and reveals that the HOOI achieves the optimal performance both statistically and computationally. Allowing the noise to be heteroskedastic, Han et al. (2022b) shows that the optimal error rate can be achieved by the projected gradient descent with the initialization given by the HeteroPCA if the condition number of the true tensor is bounded. In contrast to the prior literature, we consider the tensor and subspace estimation problem under heteroskedastic noise and aim to accommodate an arbitrarily large condition number; we show that the HOOI algorithm initialized by Deflated-HeteroPCA yields optimal theoretical guarantees. In addition to the Tucker-rank decomposition, the tensor PCA/SVD model with the low CP-rank structure (Kolda and Bader, 2009; Anandkumar et al., 2014; Cai et al., 2021, 2022a, 2023) and the low tensor-train-rank structure (Zhou et al., 2022; Cai et al., 2022b) have also received much attention in the past few years.

In addition, recent years have witnessed much acitivity in developing ℓ_{∞} and $\ell_{2,\infty}$ theoretical guarantees for singular subspaces and eigenspaces (Zhong and Boumal, 2018; Fan et al., 2018; Cape et al., 2019; Agterberg et al., 2022). Particularly worth noting is the leave-one-out analysis framework, which emerges as a powerful tool to derive fine-grained (e.g., entrywise or rowwise) bounds and finds applications in numerous high-dimensional estimation problems (Zhong and Boumal, 2018; Ma et al., 2020; Chen et al., 2019a; Abbe et al., 2020; Chen et al., 2020, 2019b, 2021c; Cai et al., 2021; Chen et al., 2021d; Cai et al., 2022a; Abbe et al., 2022; Yan et al., 2021; Ling, 2022; Zhang and Zhou, 2022; Yang and Ma, 2022). However, existing $\ell_{2,\infty}$ estimation guaranteed obtained by means of the leave-one-out technique still rely on the condition number. To achieve a condition-number-free $\ell_{2,\infty}$ bound, we provide a novel analysis based on the representation theorem presented in Xia (2021). The idea also shares similar spirit with the Neumann trick, which is commonly used in ℓ_{∞} eigenvector analysis (Eldridge et al., 2018; Chen et al., 2021a; Cheng et al., 2021).

8 Discussion

This paper has studied subspace estimation from noisy low-rank matrices in the presence of unbalanced dimensionality and heteroskedastic noise. Recognizing a curse of ill-conditioning that appears in two cuttingedge algorithms, we have developed a new algorithm called Deflated-HeteroPCA to strengthen the state-ofthe-art statistical performance in the face of a large condition number, without compromising the range of SNRs that can be accommodated. We have demonstrated that the proposed estimator enjoys nearly rateoptimal statistical guarantees (in terms of both the spectral-norm error and the more fine-grained $\ell_{2,\infty}$ -based error), which are unaffected by the underlying condition number (regardless of how large it is). When applied to two concrete statistical models (i.e., factor models and tensor PCA), our theory has led to remarkable improvement over the prior art (particularly for the ill-conditioned scenarios).

Our work suggests several potential avenues for future investigation. For example, the signal-to-noise ratio conditions (15a) and (20a) in our theory remain sub-optimal when it comes to their dependency on the rank r. How to tighten this rank dependency calls for a more refined analysis or a more powerful algorithm. Another direction worthy of future studies is the case with missing data (i.e., suppose we only have access to highly incomplete observations of the entries of the data matrix \mathbf{Y} in (1)). It would be of great interest to extend our approach and develop a computationally efficient estimator that enjoys condition-number-free and rate-optimal estimation guarantees in the presence of missing data.

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A Proof of Theorem 1 (ℓ_2 analysis for Deflated-HeteroPCA)

Before continuing, we introduce some notation about some intermediate objects that appear in our algorithm, which will be useful in the proofs. First, set

$$\boldsymbol{G}_{k+1}^0 := \boldsymbol{G}_k, \qquad 0 \le k \le k_{\max}, \tag{43a}$$

where we recall that

$$G_0 = \mathcal{P}_{\mathsf{off} ext{-}\mathsf{diag}}(oldsymbol{Y}oldsymbol{Y}^+)$$

For each $t = 0, 1, ..., t_{k+1}$ and $k = 0, 1, ..., k_{max}$, let

$$\boldsymbol{U}_{k+1}^{t}\boldsymbol{\Lambda}_{k+1}^{t}\boldsymbol{U}_{k+1}^{t\top} := \text{ the rank-}r_{k} \text{ leading eigendecomposition of } \boldsymbol{G}_{k+1}^{t},$$
(43b)

and define

$$\boldsymbol{G}_{k+1}^{t+1} := \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{G}_{k+1}^{t}\right) + \mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{U}_{k+1}^{t}\boldsymbol{\Lambda}_{k+1}^{t}\boldsymbol{U}_{k+1}^{t\top}\right),\tag{43c}$$

which corresponds to the matrix computed by HeteroPCA in the t-th iteration of the (k + 1)-th round.

In this section, we intend to prove a slightly more general version of Theorem 1 as follows.

Theorem 3. Suppose that Assumption 1 holds. Suppose that

$$\sigma_r^{\star} \ge C_0 r \sqrt{\mu r \omega_{\max}^2 + \omega_{\mathsf{col}} \left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}}\right)} \sqrt{\log n} \tag{44a}$$

$$\mu \le c_0 \frac{n_1}{r^3} \tag{44b}$$

for some sufficiently large (resp. small) constant $C_0 > 0$ (resp. $c_0 > 0$). If the numbers of iterations obey (16a), then with probability exceeding $1 - O(n^{-10})$, the output returned by Algorithm 2 satisfies

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \lesssim \frac{\sqrt{(\mu r \omega_{\max}^2 + \omega_{\text{col}}^2)\log n}}{\sigma_r^{\star}} + \frac{\omega_{\text{col}}\omega_{\text{row}}\log n}{\sigma_r^{\star 2}} + e^{-t_{k_{\max}}}.$$
(45)

Evidently, if we further have $0 < \mu r \omega_{\max}^2 \leq \omega_{col}^2$ and if the number of iterations $t_{k_{\max}}$ obeys (16b), then it is easy to check that the bound (45) (resp. the signal-to-noise ratio condition (44a)) implies (17) (resp. (15a)). This allows us to focus attention on establishing Theorem 3.

A.1 A key intermediate result and the proof of Theorem 3

Towards proving Theorem 3, we first single out a deterministic result that plays a crucial role in bounding $||UR_U - U^*||$; its proof is postponed to Section A.2.

Theorem 4. Suppose that we observe a matrix $M = \overline{U} \overline{\Lambda} \overline{U}^{\top} + Z$, where $\overline{\Lambda} \in \mathbb{R}^{r \times r}$ is a diagonal matrix with diagonal entries $\overline{\lambda}_1 \geq \cdots \geq \overline{\lambda}_r > 0$ and $\overline{U} \in \mathcal{O}^{n_1,r}$ satisfies

$$\|\overline{\boldsymbol{U}}\|_{2,\infty} \le \sqrt{\frac{\overline{\mu}r}{n_1}} \qquad with \ \overline{\mu} \le c_0 \frac{n_1}{r^3} \tag{46a}$$

for some sufficiently small constant $c_0 > 0$. Also, assume that

$$\overline{\lambda}_r \ge C_0 r \|\mathcal{P}_{\mathsf{off-diag}}(\boldsymbol{Z})\|. \tag{46b}$$

Then Algorithm 2 with initialization $G_0 = \mathcal{P}_{off-diag}(M)$ yields an estimate U satisfying

$$\left\| \boldsymbol{U}\boldsymbol{U}^{\top} - \overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}^{\top} \right\| \lesssim \frac{\left\| \mathcal{P}_{\mathsf{off}\text{-diag}}\left(\boldsymbol{Z}\right) \right\|}{\overline{\lambda}_{r}} + e^{-t_{k_{\mathsf{max}}}},\tag{47}$$

provided that the numbers of iterations obey

$$t_1 > \log\left(\frac{\frac{\overline{\mu}r}{n_1}\overline{\lambda}_1}{\sqrt{\frac{\overline{\mu}r}{n_1}}\overline{\lambda}_{r_1+1} + \|\mathcal{P}_{\mathsf{off}\text{-diag}}\left(\boldsymbol{Z}\right)\|}\right) \vee 0 \tag{48a}$$

$$t_{k} > \log\left(\frac{7\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{k-1}+1} + \|\mathcal{P}_{\mathsf{off}\operatorname{-diag}}\left(\boldsymbol{Z}\right)\|}{\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{k}+1} + \|\mathcal{P}_{\mathsf{off}\operatorname{-diag}}\left(\boldsymbol{Z}\right)\|}\right), \quad 1 \le k \le k_{\mathsf{max}} - 1.$$

$$(48b)$$

In a nutshell, Theorem 4 asserts that the subspace estimation error of Deflated-HeteroPCA depends only on (i) the size of Z after diagonal deletion and (ii) the *r*-th leading eigenvalue of $\overline{U} \overline{\Lambda} \overline{U}^{\top}$, provided that the numbers of iterations exceed some logarithmic factors. Notably, the estimation error bound (47) holds irrespective of the condition number of $\overline{\Lambda}$ and the noise entries $\mathcal{P}_{diag}(Z)$ in the diagonal (in fact, these diagonal entries of Z are never used by Deflated-HeteroPCA).

We now demonstrate how to invoke Theorem 4 to establish Theorem 3, which consists of several steps below. Before proceeding, we isolate one important matrix $U^* \Sigma^* + EV^* \in \mathbb{R}^{n_1 \times r}$, and denote its SVD as

$$\widetilde{U}\widetilde{\Sigma}\widetilde{W}^{\top} = U^{\star}\Sigma^{\star} + EV^{\star}, \tag{49}$$

where $\widetilde{U} \in \mathcal{O}^{n_1,r}, \widetilde{W} \in \mathcal{O}^{r,r}$ and $\widetilde{\Sigma} = \operatorname{diag}(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_r)$ with $\widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_r \geq 0$.

Step 1: bounding the spectrum of $\tilde{\Sigma}^{-1}$. We start by controlling the spectrum of $\tilde{\Sigma}$. Taking Weyl's inequality, Assumption 1 and Lemma 5 together implies that with probability exceeding $1 - O(n^{-10})$,

$$\begin{split} \max_{1 \le i \le r} |\widetilde{\sigma}_i - \sigma_i^{\star}| \le \| \boldsymbol{E} \boldsymbol{V}^{\star} \| \lesssim \sqrt{(r\omega_{\max}^2 + \omega_{\operatorname{col}}^2)\log n} + B\log n \sqrt{\frac{\mu_2 r}{n_2}} \\ \lesssim \sqrt{(r\omega_{\max}^2 + \omega_{\operatorname{col}}^2)\log n} + \frac{\omega_{\operatorname{row}}}{\sqrt{\log n}}\log n \sqrt{\frac{\mu_2 r}{n_2}} \\ \lesssim \sqrt{(r\omega_{\max}^2 + \omega_{\operatorname{col}}^2)\log n} + \sqrt{n_2}\omega_{\max}\sqrt{\log n} \sqrt{\frac{\mu r}{n_2}} \end{split}$$

$$\lesssim \sqrt{(\mu r \omega_{\max}^2 + \omega_{col}^2) \log n},\tag{50}$$

where the second line relies on Assumption 1. Consequently, one can deduce that

$$\left\|\widetilde{\boldsymbol{\Sigma}}^{-1}\right\| = \frac{1}{\widetilde{\sigma}_r} \le \frac{1}{\sigma_r^{\star} - \left\|\boldsymbol{E}\boldsymbol{V}^{\star}\right\|} \le \frac{\sqrt{2}}{\sigma_r^{\star}},\tag{51}$$

provided that $\sigma_r^{\star} \geq C_0 \sqrt{(\mu r \omega_{\max}^2 + \omega_{col}^2) \log n}$ for some large enough constant $C_0 > 0$. It is also seen that

$$\sigma_r^{\star} \leq \widetilde{\sigma}_r + \| \boldsymbol{E} \boldsymbol{V}^{\star} \| \leq \widetilde{\sigma}_r + \frac{1}{2} \sigma_r^{\star}$$

$$\implies \quad \sigma_r^{\star} \leq 2 \widetilde{\sigma}_r.$$
(52)

Repeating the same argument also reveals that

$$\frac{1}{2}\widetilde{\sigma}_i \le \sigma_i^* \le 2\widetilde{\sigma}_i, \qquad 1 \le i \le r.$$
(53)

Step 2: bounding $\|\tilde{U}\|_{2,\infty}$. We now move on to control $\|\tilde{U}\|_{2,\infty}$, a sort of incoherence condition needed in order to invoke Theorem 4 (see (46a)). Towards this, we would like to first the discrepancy between $U^*U^{*\top}\tilde{U}$ and \tilde{U} , which would in turn allow us to switch attention to the $\ell_{2,\infty}$ norm of $U^*U^{*\top}\tilde{U}$. Recognizing that

$$(\widetilde{U} - U^{\star}U^{\star\top}\widetilde{U})\widetilde{\Sigma}\widetilde{W}^{\top} = \mathcal{P}_{U_{\perp}^{\star}}\widetilde{U}\widetilde{\Sigma}\widetilde{W}^{\top} = \mathcal{P}_{U_{\perp}^{\star}}(U^{\star}\Sigma^{\star} + EV^{\star})$$

= $EV^{\star} - U^{\star}U^{\star\top}EV^{\star},$ (54)

we can readily use $\|\widetilde{\boldsymbol{W}}\| = 1$ to derive

$$\begin{aligned} \left\| \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}} \right\|_{2,\infty} &= \left\| \left(\boldsymbol{E} \boldsymbol{V}^{\star} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right) \widetilde{\boldsymbol{W}} \widetilde{\boldsymbol{\Sigma}}^{-1} \right\|_{2,\infty} \\ &\leq \left(\left\| \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} + \left\| \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \right) \left\| \widetilde{\boldsymbol{\Sigma}}^{-1} \right\|. \end{aligned}$$
(55)

In view of Lemma 5 and Assumption 1, with probability exceeding $1 - O(n^{-10})$, one has

$$\left\| \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \lesssim \left(B \log n + \omega_{\mathsf{row}} \sqrt{\log n} \right) \sqrt{\frac{\mu_2 r}{n_2}} \asymp \omega_{\mathsf{row}} \sqrt{\log n} \sqrt{\frac{\mu_2 r}{n_2}}$$

and

$$\begin{split} \left\| \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} &\leq \left\| \boldsymbol{U}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \\ &\lesssim \sqrt{\frac{\mu_1 r}{n_1}} \left(B \frac{\mu r}{\sqrt{n_1 n_2}} \log n + \left(\sqrt{\frac{\mu r}{n_2}} \omega_{\mathsf{row}} + \sqrt{\frac{\mu r}{n_1}} \omega_{\mathsf{col}} \right) \sqrt{\log n} \right) \\ &\lesssim \sqrt{\frac{\mu_1 r}{n_1}} \left(\frac{\omega_{\mathsf{row}}}{\sqrt{\log n}} \sqrt{\frac{\mu r}{n_2}} \log n + \left(\sqrt{\frac{\mu r}{n_2}} \omega_{\mathsf{row}} + \sqrt{\frac{\mu r}{n_1}} \omega_{\mathsf{col}} \right) \sqrt{\log n} \right) \\ &\lesssim \sqrt{\frac{\mu_1 r}{n_1}} \left(\sqrt{\frac{\mu r}{n_2}} \omega_{\mathsf{row}} + \sqrt{\frac{\mu r}{n_1}} \omega_{\mathsf{col}} \right) \sqrt{\log n}, \end{split}$$

where the second line has also made use of the assumption that $\mu r \leq n_1$. Putting (55) and the previous two inequalities together and using the assumption $\mu r \leq n_1$, we arrive at

$$\begin{split} \left\| \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}} \right\|_{2,\infty} &\lesssim \left(\omega_{\mathsf{row}} \sqrt{\log n} \sqrt{\frac{\mu_2 r}{n_2}} + \sqrt{\frac{\mu_1 r}{n_1}} \left(\sqrt{\frac{\mu r}{n_2}} \omega_{\mathsf{row}} + \sqrt{\frac{\mu r}{n_1}} \omega_{\mathsf{col}} \right) \sqrt{\log n} \right) \frac{1}{\sigma_r^{\star}} \\ &\lesssim \frac{\left(\sqrt{\frac{\mu r}{n_2}} \omega_{\mathsf{row}} + \frac{\mu r}{n_1} \omega_{\mathsf{col}} \right) \sqrt{\log n}}{\sigma_r^{\star}} \end{split}$$

$$\ll \frac{1}{r} = \sqrt{\frac{\left(\frac{n_1}{r^3}\right)r}{n_1}} \tag{56}$$

with probability exceeding $1 - O(n^{-10})$, provided that

$$\sigma_r^{\star} \gg r \left[\left(\sqrt{\frac{\mu r}{n_2}} \omega_{\rm row} + \frac{\mu r}{n_1} \omega_{\rm col} \right) \sqrt{\log n} + \sqrt{(\mu r \omega_{\rm max}^2 + \omega_{\rm col}^2) \log n} \right] \asymp r \sqrt{(\mu r \omega_{\rm max}^2 + \omega_{\rm col}^2) \log n}.$$

As a result, with probability at least $1 - O(n^{-10})$, we reach the following upper bound:

$$\begin{aligned} \|\widetilde{\boldsymbol{U}}\|_{2,\infty} &\leq \|\boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}}\|_{2,\infty} + \|\boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}}\|_{2,\infty} \\ &\leq \|\boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}}\|_{2,\infty} + \|\boldsymbol{U}^{\star}\|_{2,\infty} \|\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}}\| \\ &\leq \|\boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}}\|_{2,\infty} + \sqrt{\frac{\mu r}{n_1}} \ll \sqrt{\frac{\left(\frac{n_1}{r^3}\right)r}{n_1}}, \end{aligned}$$
(57)

where the last inequality holds under our assumption that $\mu r^3 \leq n_1$. With this $\ell_{2,\infty}$ bound for \tilde{U} in place — which reveals an upper bound $O(\frac{n_1}{r^3})$ on the incoherence parameter of \tilde{U} (see the requirement (46a)) — we can proceed to apply Theorem 4 in the next step.

Step 3: bounding $\|UU^{\top} - U^{\star}U^{\star \top}\|$ and $\|UR_U - U^{\star}\|$. In this step, we shall first invoke Theorem 4 to control $\|UU^{\top} - \widetilde{U}\widetilde{U}^{\top}\|$, and then apply standard eigenspace perturbation theory to bound $\|\widetilde{U}\widetilde{U}^{\top} - U^{\star}U^{\star \top}\|$.

To begin with, let us write

$$\boldsymbol{Y}\boldsymbol{Y}^{\top} = \left(\boldsymbol{X}^{\star} + \boldsymbol{E}\right)\left(\boldsymbol{X}^{\star} + \boldsymbol{E}\right)^{\top} = \left(\boldsymbol{U}^{\star}\boldsymbol{\Sigma}^{\star} + \boldsymbol{E}\boldsymbol{V}^{\star}\right)\left(\boldsymbol{U}^{\star}\boldsymbol{\Sigma}^{\star} + \boldsymbol{E}\boldsymbol{V}^{\star}\right)^{\top} + \left(\boldsymbol{E}\boldsymbol{E}^{\top} - \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top}\right).$$
(58)

Recall that \widetilde{U} represents the column subspace of $(U^*\Sigma^* + EV^*) (U^*\Sigma^* + EV^*)^\top$ (cf. (49)). Thus, in order to apply Theorem 4 to control $||UU^\top - \widetilde{U}\widetilde{U}^\top||$, the key lies in coping with $||\mathcal{P}_{off-diag}(EE^\top - EV^*V^*^\top E^\top)||$. By virtue of Lemma 7 and Assumption 1, with probability exceeding $1 - O(n^{-10})$ we have

$$\begin{aligned} \left\| \mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right\| &\lesssim B^2 \log^2 n + \omega_{\mathsf{col}} \left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}} \right) \log n \\ &\lesssim \frac{\omega_{\mathsf{row}} \omega_{\mathsf{col}}}{\log n} \log^2 n + \omega_{\mathsf{col}} \left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}} \right) \log n \\ &\asymp \omega_{\mathsf{col}} \left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}} \right) \log n. \end{aligned}$$
(59)

Putting (50) and (59) together, we arrive at, with probability exceeding $1 - O(n^{-10})$,

$$\begin{split} \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} - \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top} \right) \right\| &\leq \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right\| + \left\| \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top} \right\| + \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top} \right) \right\| \\ &\leq \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right\| + 2 \left\| \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top} \right\| \\ &\leq \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right\| + 2 \left\| \boldsymbol{E}\boldsymbol{V}^{\star} \right\|^{2} \\ &\lesssim \omega_{\mathsf{col}} \left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}} \right) \log n + \left(\mu r \omega_{\mathsf{max}}^{2} + \omega_{\mathsf{col}}^{2} \right) \log n \\ &\asymp \left(\mu r \omega_{\mathsf{max}}^{2} + \omega_{\mathsf{col}} \left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}} \right) \right) \log n \\ &\ll \frac{\sigma_{r}^{\star 2}}{r} \lesssim \frac{\widetilde{\sigma}_{r}^{2}}{r}, \end{split}$$

where the last inequality arises from our assumption (44a) on σ_r^* and (52). In view of Theorem 4, (52), (57) and the previous inequality, we can easily check that: if $\{t_i\}$ satisfy (16a), then one has

$$\|\boldsymbol{U}\boldsymbol{U}^{\top} - \widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{U}}^{\top}\| \leq \frac{\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top} - \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top}\right)\|}{\widetilde{\sigma}_{r}^{2}} + e^{-t_{k_{\mathsf{max}}}} \\ \lesssim \frac{\left(\mu r \omega_{\mathsf{max}}^{2} + \omega_{\mathsf{col}}\left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}}\right)\right)\log n}{\sigma_{r}^{\star 2}} + e^{-t_{k_{\mathsf{max}}}}$$
(60)

with probability exceeding $1 - O(n^{-10})$, provided that

$$\begin{split} \sigma_r^{\star} \gg r \left[\sqrt{(\mu r \omega_{\max}^2 + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) \right)} + \sqrt{\frac{\mu r}{n_2}} \omega_{\text{row}} + \frac{\mu r}{n_1} \omega_{\text{col}} \right] \sqrt{\log n} \\ \approx r \sqrt{(\mu r \omega_{\max}^2 + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) \right) \log n}. \end{split}$$

Next, let us turn to bounding $\|\widetilde{U}\widetilde{U}^{\top} - U^{\star}U^{\star\top}\|$. Taking (50) and the sin Θ theorem (Chen et al., 2021b, Theorem 2.9) together shows that

$$\left\| \widetilde{\boldsymbol{U}} \widetilde{\boldsymbol{U}}^{\top} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \right\| \lesssim \frac{\sqrt{(\mu r \omega_{\max}^2 + \omega_{\mathsf{col}}^2) \log n}}{\sigma_r^* - \| \boldsymbol{E} \boldsymbol{V}^{\star} \|} \asymp \frac{\sqrt{(\mu r \omega_{\max}^2 + \omega_{\mathsf{col}}^2) \log n}}{\sigma_r^{\star}}$$

with probability at least $1 - O(n^{-10})$. Combine this with (60) and invoke the triangle inequality to yield

$$\begin{split} \left\| \boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top} \right\| &\leq \left\| \boldsymbol{\widetilde{U}}\boldsymbol{\widetilde{U}}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top} \right\| + \left\| \boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{\widetilde{U}}\boldsymbol{\widetilde{U}}^{\top} \right\| \\ &\lesssim \frac{\sqrt{(\mu r \omega_{\max}^2 + \omega_{col}^2)\log n}}{\sigma_r^{\star}} + \frac{(\mu r \omega_{\max}^2 + \omega_{col} \left(\omega_{\operatorname{row}} + \omega_{col}\right))\log n}{\sigma_r^{\star^2}} + e^{-t_{k_{\max}}} \\ &\approx \frac{\sqrt{(\mu r \omega_{\max}^2 + \omega_{col}^2)\log n}}{\sigma_r^{\star}} + \left(\frac{\sqrt{(\mu r \omega_{\max}^2 + \omega_{col}^2)\log n}}{\sigma_r^{\star}} \right)^2 + \frac{\omega_{col}\omega_{\operatorname{row}}\log n}{\sigma_r^{\star^2}} + e^{-t_{k_{\max}}} \\ &\approx \frac{\sqrt{(\mu r \omega_{\max}^2 + \omega_{col}^2)\log n}}{\sigma_r^{\star}} + \frac{\omega_{col}\omega_{\operatorname{row}}\log n}{\sigma_r^{\star^2}} + e^{-t_{k_{\max}}} \end{split}$$

under our assumption on σ_r^{\star} . Finally, using the basic inequality $\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \leq \sqrt{2} \|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\|$ (Chen et al., 2021b, Lemma 2.5) yields the desired result in Theorem 3.

To finish up, it suffices to justify the intermediate result in Theorem 4, which we shall accomplish next.

A.2 Proof of Theorem 4

We now present our proof of Theorem 4. Recall the definitions of G_k^t and U_k^t in (43a)-(43c). For any $k \ge 1$ and $0 \le t \le t_k$, we introduce the following convenient notation:

$$\overline{\boldsymbol{M}} = \overline{\boldsymbol{U}} \,\overline{\boldsymbol{\Lambda}} \,\overline{\boldsymbol{U}}^{\top}, \quad D_k^t = \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{G}_k^t - \overline{\boldsymbol{M}} \right) \right\|, \quad L_k^t = \left\| \boldsymbol{G}_k^t - \overline{\boldsymbol{M}} \right\|, \quad and \quad \overline{\boldsymbol{U}}_k = \overline{\boldsymbol{U}}_{:,1:r_k}. \tag{61}$$

Step 1: a basic property about r_1 as selected in Algorithm 2. For k = 1, we first show that the rank r_1 selected in Algorithm 2 lies within

$$r_{1} \in \mathcal{R}_{1} := \left\{ r' \leq r : \frac{\sigma_{1}\left(\boldsymbol{G}_{0}\right)}{\sigma_{r'}\left(\boldsymbol{G}_{0}\right)} \leq 4 \quad \text{and} \quad \sigma_{r'}\left(\boldsymbol{G}_{0}\right) - \sigma_{r'+1}\left(\boldsymbol{G}_{0}\right) \geq \frac{1}{r}\sigma_{r'}\left(\boldsymbol{G}_{0}\right) \right\}.$$
(62)

To do so, it suffices to verify that \mathcal{R}_1 is non-empty, towards which we divide into two scenarios.

• Case 1: $\{i \in [r-1] : \sigma_i(\mathbf{G}_0) \geq \frac{r}{r-1}\sigma_{i+1}(\mathbf{G}_0)\}$ is non-empty. Take $1 \leq \tilde{r} \leq r-1$ to be the smallest entry in this set. Then it is seen that

$$\frac{\sigma_1\left(\boldsymbol{G}_0\right)}{\sigma_{\widetilde{r}}\left(\boldsymbol{G}_0\right)} = \prod_{j=1}^{\widetilde{r}-1} \frac{\sigma_j\left(\boldsymbol{G}_0\right)}{\sigma_{j+1}\left(\boldsymbol{G}_0\right)} \le \left(\frac{r}{r-1}\right)^{r-2} \le 4,\tag{63}$$

thus implying that $\tilde{r} \in \mathcal{R}_1$.

• Case 2: $\{i \in [r-1] : \sigma_i(\mathbf{G}_0) \geq \frac{r}{r-1}\sigma_{i+1}(\mathbf{G}_0)\}$ is empty. In this case, one necessarily has

$$\frac{\sigma_1\left(\boldsymbol{G}_0\right)}{\sigma_r\left(\boldsymbol{G}_0\right)} \le \left(\frac{r}{r-1}\right)^{r-1} < e < 4.$$

By virtue of the definition $\boldsymbol{G}_1^0 = \boldsymbol{G}_0 = \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\overline{\boldsymbol{U}}\,\overline{\boldsymbol{\Lambda}}\,\overline{\boldsymbol{U}}^\top + \boldsymbol{Z})$ (see (43a)), one can derive

$$L_{1}^{0} = \left\| \mathcal{P}_{\mathsf{diag}} \left(\overline{\boldsymbol{U}} \, \overline{\boldsymbol{\Lambda}} \, \overline{\boldsymbol{U}}^{\top} \right) - \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{Z} \right) \right\|$$

$$\leq \left\| \mathcal{P}_{\mathsf{diag}} \left(\overline{\boldsymbol{U}} \, \overline{\boldsymbol{\Lambda}} \, \overline{\boldsymbol{U}}^{\top} \right) \right\| + \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{Z} \right) \right\|$$

$$\leq \left\| \overline{\boldsymbol{U}} \right\|_{2,\infty}^{2} \left\| \overline{\boldsymbol{\Lambda}} \right\| + \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{Z} \right) \right\| \leq \frac{\overline{\mu}r}{n_{1}} \overline{\lambda}_{1} + \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{Z} \right) \right\|.$$
(64)

Weyl's inequality then reveals that, for all $i \in [n_1]$,

$$\left|\overline{\lambda}_{i} - \sigma_{i}\left(\boldsymbol{G}_{0}\right)\right| \leq L_{1}^{0} \leq \frac{\overline{\mu}r}{n_{1}}\overline{\lambda}_{1} + \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\|,\tag{65}$$

which together with the assumptions (46a) and (46b) immediately tells us that

$$\sigma_1\left(\boldsymbol{G}_0\right) \ge \left(1 - \frac{\overline{\mu}r}{n_1}\right)\overline{\lambda}_1 - \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\| \ge \left(1 - \frac{\overline{\mu}r}{n_1}\right)\overline{\lambda}_1 - \frac{\overline{\lambda}_1}{C_0r} \ge \frac{1}{2}\overline{\lambda}_1.$$

Combining (64) and (65) with the assumptions (46a) and (46b) also leads to

$$\sigma_{r} (\boldsymbol{G}_{0}) - \sigma_{r+1} (\boldsymbol{G}_{0}) \geq \sigma_{r} (\boldsymbol{G}_{0}) - L_{1}^{0}$$

$$\geq \sigma_{r} (\boldsymbol{G}_{0}) - \left(\frac{\overline{\mu}r}{n_{1}}\overline{\lambda}_{1} + \|\mathcal{P}_{\text{off-diag}} (\boldsymbol{Z})\|\right)$$

$$\geq \sigma_{r} (\boldsymbol{G}_{0}) - \left(\frac{\overline{\mu}r}{n_{1}}\overline{\lambda}_{1} + \frac{\overline{\lambda}_{1}}{C_{0}r}\right)$$

$$\geq \sigma_{r} (\boldsymbol{G}_{0}) - \frac{1}{8}\sigma_{1} (\boldsymbol{G}_{0})$$

$$\geq \frac{1}{2}\sigma_{r} (\boldsymbol{G}_{0}) \geq \frac{1}{r}\sigma_{r} (\boldsymbol{G}_{0}) . \qquad (66)$$

Putting (63) and (66) for the above two cases together confirms that $\mathcal{R}_1 \neq \emptyset$, and hence (62) is always true.

Step 2: bounding $L_1^t = \|\boldsymbol{G}_1^t - \overline{\boldsymbol{M}}\|$. Next, we look at the difference between the iterate \boldsymbol{G}_1^t (in the first round) and the low-rank matrix $\overline{\boldsymbol{M}}$. We will prove by induction the two properties below: for all $t \ge 0$,

$$\overline{\lambda}_{r_1} \ge 18rL_1^t, \tag{67a}$$

$$L_{1}^{t} - 6\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{1}+1} - 4 \left\| \mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right) \right\| \leq \frac{1}{e^{t}} \left(L_{1}^{0} - 6\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{1}+1} - 4 \left\| \mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right) \right\| \right).$$
(67b)

Step 2.1: the base case for (67a) and (67b). Let us start with the base case with t = 0. Noting that (64) and (65) hold and recalling that $\sigma_1(\mathbf{G}_0)/\sigma_{r_1}(\mathbf{G}_0) \leq 4$ and $\overline{\lambda}_1 \geq \overline{\lambda}_{r_1} \geq \overline{\lambda}_r \gg r \|\mathcal{P}_{\mathsf{off}-\mathsf{diag}}(\mathbf{Z})\|$, we have

$$\begin{split} L_{1}^{0} &\leq \overline{\frac{\mu}{n_{1}}}\overline{\lambda}_{1} + \left\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right)\right\| \leq \overline{\frac{\mu}{n_{1}}}\left(\sigma_{1}\left(\boldsymbol{G}_{0}\right) + L_{1}^{0}\right) + \left\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right)\right\| \\ &\leq \overline{\frac{\mu}{n_{1}}}\left(4\sigma_{r_{1}}\left(\boldsymbol{G}_{0}\right) + L_{1}^{0}\right) + \left\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right)\right\| \\ &\leq \overline{\frac{\mu}{n_{1}}}\left[4\overline{\lambda}_{r_{1}} + 5L_{1}^{0}\right] + \left\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right)\right\| \\ &\leq \overline{\frac{\lambda}{r_{1}}} + \frac{1}{2}L_{1}^{0} + \frac{\overline{\lambda}_{r_{1}}}{72r} = \frac{\overline{\lambda}_{r_{1}}}{36r} + \frac{1}{2}L_{1}^{0}, \end{split}$$

where the last line also makes use of the assumptions (46a) and (46b). This further tells us that

$$\overline{\lambda}_{r_1} \ge 18rL_1^0,$$

as claimed in (67a) when t = 0. Combining Weyl's inequality, (65), and the previous inequality gives

$$\overline{\lambda}_{r_{1}} - \overline{\lambda}_{r_{1}+1} \geq \sigma_{r_{1}} (\boldsymbol{G}_{0}) - \sigma_{r_{1}+1} (\boldsymbol{G}_{0}) - \left| \sigma_{r_{1}} (\boldsymbol{G}_{0}) - \overline{\lambda}_{r_{1}} \right| - \left| \sigma_{r_{1}+1} (\boldsymbol{G}_{0}) - \overline{\lambda}_{r_{1}+1} \right| \\
\geq \frac{1}{r} \sigma_{r_{1}} (\boldsymbol{G}_{0}) - 2L_{1}^{0} \geq \frac{1}{r} \left(\overline{\lambda}_{r_{1}} - \left| \sigma_{r_{1}} (\boldsymbol{G}_{0}) - \overline{\lambda}_{r_{1}} \right| \right) - 2L_{1}^{0} \\
\geq \frac{\overline{\lambda}_{r_{1}}}{r} - 3L_{1}^{0} \geq \frac{3\overline{\lambda}_{r_{1}}}{4r} \lor 9L_{1}^{0}.$$
(68)

The inequality (67b) for the base case with t = 0 holds trivially.

Step 2.2: induction step for (67a) and (67b). Now, supposing that (67a) and (67b) hold for t - 1, we would like to justify these two claims for t. In light of Algorithm 1, we first observe that

$$\left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{G}_{1}^{t}-\overline{\boldsymbol{M}}\right)\right\| = \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{G}_{0}-\overline{\boldsymbol{M}}\right)\right\| = \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\|$$
(69)

and

$$\begin{aligned} \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{G}_{1}^{t} - \overline{\boldsymbol{M}} \right) \right\| &= \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{P}_{\boldsymbol{U}_{1}^{t-1}} \boldsymbol{G}_{1}^{t-1} - \overline{\boldsymbol{M}} \right) \right\| \\ &\leq \underbrace{\left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{P}_{\overline{\boldsymbol{U}}_{1}} \left(\boldsymbol{G}_{1}^{t-1} - \overline{\boldsymbol{M}} \right) \right) \right\|}_{=:\alpha_{1}} + \underbrace{\left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{P}_{\left(\boldsymbol{U}_{1}^{t-1}\right)_{\perp}} \overline{\boldsymbol{M}} \right) \right\|}_{=:\alpha_{2}} + \underbrace{\left\| \mathcal{P}_{\mathsf{diag}} \left(\left(\boldsymbol{P}_{\boldsymbol{U}_{1}^{t-1}} - \boldsymbol{P}_{\overline{\boldsymbol{U}}_{1}} \right) \left(\boldsymbol{G}_{1}^{t-1} - \overline{\boldsymbol{M}} \right) \right) \right\|}_{=:\alpha_{3}}. \end{aligned}$$
(70)

• In view of Zhang et al. (2022, Lemma 1), one can upper bound the first term α_1 as

$$\alpha_1 \le \sqrt{\frac{\overline{\mu}r}{n_1}} \left\| \boldsymbol{G}_1^{t-1} - \overline{\boldsymbol{M}} \right\| = \sqrt{\frac{\overline{\mu}r}{n_1}} L_1^{t-1}.$$
(71)

• Turning to α_2 , applying Zhang et al. (2022, Lemma 1) again yields

$$\begin{aligned} \alpha_{2} &= \left\| \mathcal{P}_{\mathsf{diag}} \left(\mathbf{P}_{\left(\mathbf{U}_{1}^{t-1} \right)_{\perp}} \overline{\mathbf{M}} \right) \right\| = \left\| \mathcal{P}_{\mathsf{diag}} \left(\mathbf{P}_{\left(\mathbf{U}_{1}^{t-1} \right)_{\perp}} \overline{\mathbf{M}} \mathbf{P}_{\overline{\mathbf{U}}} \right) \right\| \leq \sqrt{\frac{\overline{\mu}r}{n_{1}}} \left\| \mathbf{P}_{\left(\mathbf{U}_{1}^{t-1} \right)_{\perp}} \overline{\mathbf{M}} \right\| \\ &\leq \sqrt{\frac{\overline{\mu}r}{n_{1}}} \left(\left\| \mathbf{P}_{\left(\mathbf{U}_{1}^{t-1} \right)_{\perp}} \left(\mathbf{P}_{\overline{\mathbf{U}}_{1}} \overline{\mathbf{M}} \right) \right\| + \left\| \mathbf{P}_{\left(\overline{\mathbf{U}}_{1} \right)_{\perp}} \overline{\mathbf{M}} \right\| \right) \\ &= \sqrt{\frac{\overline{\mu}r}{n_{1}}} \left(\left\| \mathbf{P}_{\left(\mathbf{U}_{1}^{t-1} \right)_{\perp}} \left(\mathbf{P}_{\overline{\mathbf{U}}_{1}} \overline{\mathbf{M}} \right) \right\| + \overline{\lambda}_{r_{1}+1} \right), \end{aligned}$$

where the second identity is valid since \overline{M} falls within the subspace \overline{U} . Recognizing that

$$oldsymbol{G}_1^{t-1} = oldsymbol{P}_{\overline{oldsymbol{U}}_1} \overline{oldsymbol{M}} + igl(oldsymbol{G}_1^{t-1} - oldsymbol{P}_{\overline{oldsymbol{U}}_1} \overline{oldsymbol{M}}igr)$$

and

$$\left\| \boldsymbol{G}_{1}^{t-1} - \boldsymbol{P}_{\overline{\boldsymbol{U}}_{1}} \overline{\boldsymbol{M}} \right\| \leq \left\| \boldsymbol{G}_{1}^{t-1} - \overline{\boldsymbol{M}} \right\| + \left\| \boldsymbol{P}_{\left(\overline{\boldsymbol{U}}_{1}\right)_{\perp}} \overline{\boldsymbol{M}} \right\|,$$

one can invoke Lemma 8 to show that

$$\left\|\boldsymbol{P}_{\left(\boldsymbol{U}_{1}^{t-1}\right)_{\perp}}\left(\boldsymbol{P}_{\overline{\boldsymbol{U}}_{1}}\overline{\boldsymbol{M}}\right)\right\| \leq 2\left\|\boldsymbol{G}_{1}^{t-1}-\boldsymbol{P}_{\overline{\boldsymbol{U}}_{1}}\overline{\boldsymbol{M}}\right\| \leq 2\left(\left\|\boldsymbol{G}_{1}^{t-1}-\overline{\boldsymbol{M}}\right\|+\left\|\boldsymbol{P}_{\left(\overline{\boldsymbol{U}}_{1}\right)_{\perp}}\overline{\boldsymbol{M}}\right\|\right) = 2\left(\boldsymbol{L}_{1}^{t-1}+\overline{\lambda}_{r_{1}+1}\right).$$

Combining the previous two inequalities, we have

$$\alpha_2 \le \sqrt{\frac{\overline{\mu}r}{n_1}} \left(2L_1^{t-1} + 3\overline{\lambda}_{r_1+1} \right). \tag{72}$$

• Now, we move on to α_3 . Recall that U_1^{t-1} is the leading-*r* eigen-subspace of G_1^{t-1} . Combining (68), the induction hypothesis $\overline{\lambda}_{r_1} \geq 12rL_1^{t-1}$, the sin Θ Theorem (or more precisely, the perturbation bound (2.26a) in Chen et al. (2021b)) and Weyl's inequality, one has

$$\left\|\boldsymbol{P}_{\boldsymbol{U}_{1}^{t-1}}-\boldsymbol{P}_{\overline{\boldsymbol{U}}_{1}}\right\| \leq \frac{2\left\|\boldsymbol{G}_{1}^{t-1}-\overline{\boldsymbol{M}}\right\|}{\overline{\lambda}_{r_{1}}-\overline{\lambda}_{r_{1}+1}} \leq \frac{2L_{1}^{t-1}}{3\overline{\lambda}_{r_{1}}/(4r)} \leq \frac{3rL_{1}^{t-1}}{\overline{\lambda}_{r_{1}}}$$

As a consequence, one can bound α_3 as follows

$$\alpha_{3} \leq \left\| \boldsymbol{P}_{\boldsymbol{U}_{1}^{t-1}} - \boldsymbol{P}_{\overline{\boldsymbol{U}}_{1}} \right\| \left\| \boldsymbol{G}_{1}^{t-1} - \overline{\boldsymbol{M}} \right\| \leq \frac{3r \left(L_{1}^{t-1} \right)^{2}}{\overline{\lambda}_{r_{1}}}.$$
(73)

Putting (69), (70), (71), (72) and (73) together yields

$$\begin{split} L_{1}^{t} &= \left\|\boldsymbol{G}_{1}^{t} - \overline{\boldsymbol{M}}\right\| \leq \left\|\mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{Z}\right)\right\| + \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\| \leq \alpha_{1} + \alpha_{2} + \alpha_{3} + \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\| \\ &\leq 3\sqrt{\frac{\overline{\mu}r}{n_{1}}}L_{1}^{t-1} + 3\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + \frac{3r\left(L_{1}^{t-1}\right)^{2}}{\overline{\lambda}_{r_{1}}} + \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\| \\ &\leq \frac{1}{2e}L_{1}^{t-1} + 3\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + \frac{1}{2e}L_{1}^{t-1} + \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\| \\ &= \frac{1}{e}L_{1}^{t-1} + 3\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + \left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\|, \end{split}$$

where the third line holds due to the induction hypothesis (67a) for t - 1. This taken together with the induction hypothesis (67a) for t - 1 and the assumptions (46a) and (46b) implies that

$$L_1^t \le \frac{1}{e} \cdot \frac{\overline{\lambda}_{r_1}}{18r} + \frac{\overline{\lambda}_{r_1}}{72r} + \frac{\overline{\lambda}_{r_1}}{72r} \le \frac{\overline{\lambda}_{r_1}}{18r}$$

and

$$\begin{split} L_1^t - 6\sqrt{\frac{\overline{\mu}r}{n_1}}\overline{\lambda}_{r_1+1} - 4 \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\mathbf{Z}\right) \right\| &\leq \frac{1}{e} \left(L_1^{t-1} - 6\sqrt{\frac{\overline{\mu}r}{n_1}}\overline{\lambda}_{r_1+1} - 4 \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\mathbf{Z}\right) \right\| \right) \\ &\leq \frac{1}{e^t} \left(L_1^0 - 6\sqrt{\frac{\overline{\mu}r}{n_1}}\overline{\lambda}_{r_1+1} - 4 \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\mathbf{Z}\right) \right\| \right). \end{split}$$

This directly concludes the proof of (67a) and (67b) via standard induction arguments.

Step 3: bounding $L_k^t = \|\boldsymbol{G}_k^t - \overline{\boldsymbol{M}}\|$ for k > 1. Having looked at what happens in the first round, we now proceed to develop upper bounds for $\|\boldsymbol{G}_k^t - \overline{\boldsymbol{M}}\|$ when k > 1. In view of the inequality (67b), choosing the number of iterations such that $t_1 \ge \log \left(\frac{\frac{\overline{\mu}r}{n_1}\overline{\lambda}_1}{\sqrt{\frac{\overline{\mu}r}{n_1}}\overline{\lambda}_{r_1+1}+\|\mathcal{P}_{\text{off-diag}}(\boldsymbol{Z})\|}\right) \lor 0$ gives

$$L_{2}^{0} = L_{1}^{t_{1}} \leq 6\sqrt{\frac{\mu r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + 4 \left\|\mathcal{P}_{\text{off-diag}}\left(\mathbf{Z}\right)\right\| + \frac{1}{e^{t_{1}}}\left(\frac{\overline{\mu}r}{n_{1}}\overline{\lambda}_{1} + \left\|\mathcal{P}_{\text{off-diag}}\left(\mathbf{Z}\right)\right\| - 6\sqrt{\frac{\mu r}{n_{1}}}\overline{\lambda}_{r_{1}+1} - 4 \left\|\mathcal{P}_{\text{off-diag}}\left(\mathbf{Z}\right)\right\|\right) \leq 6\sqrt{\frac{\mu r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + 4 \left\|\mathcal{P}_{\text{off-diag}}\left(\mathbf{Z}\right)\right\| + \frac{1}{e^{t_{1}}} \cdot \frac{\overline{\mu}r}{n_{1}}\overline{\lambda}_{1} \leq 6\sqrt{\frac{\mu r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + 4 \left\|\mathcal{P}_{\text{off-diag}}\left(\mathbf{Z}\right)\right\| + \frac{\sqrt{\frac{\mu r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + \left\|\mathcal{P}_{\text{off-diag}}\left(\mathbf{Z}\right)\right\|}{\frac{\mu r}{n_{1}}\overline{\lambda}_{1}} \overline{\lambda}_{1} \leq 7\sqrt{\frac{\mu r}{n_{1}}}\overline{\lambda}_{r_{1}+1} + 5 \left\|\mathcal{P}_{\text{off-diag}}\left(\mathbf{Z}\right)\right\|,$$

$$(74)$$

where the first inequality results from (67b) and (65).

Similarly, setting the numbers of iterations as

$$t_k \ge \log\left(\frac{7\sqrt{\frac{\overline{\mu}r}{n_1}}\overline{\lambda}_{r_{k-1}+1} + \|\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}}\left(\mathbf{Z}\right)\|}{\sqrt{\frac{\overline{\mu}r}{n_1}}\overline{\lambda}_{r_k+1} + \|\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}}\left(\mathbf{Z}\right)\|}\right), \quad 2 \le k \le k_{\max} - 1$$

and repeating similar arguments as in (62), (67a), (67b) and (74) yield that: for all $2 \le k \le k_{\max}, t \ge 0$,

$$r_{k} \in \mathcal{R}_{k} := \left\{ r' : \frac{\sigma_{r_{k-1}+1} \left(\mathbf{G}_{k-1} \right)}{\sigma_{r'} \left(\mathbf{G}_{k-1} \right)} \le 4 \text{ and } \sigma_{r'} \left(\mathbf{G}_{k-1} \right) \ge \frac{r}{r-1} \sigma_{r'+1} \left(\mathbf{G}_{k-1} \right) \right\},$$
(75a)

$$L_{k}^{0} = L_{k-1}^{t_{k-1}} \leq 7 \sqrt{\frac{\overline{\mu}r}{n_{1}}} \overline{\lambda}_{r_{k-1}+1} + 5 \left\| \mathcal{P}_{\mathsf{off-diag}} \left(\boldsymbol{Z} \right) \right\|,$$
(75b)

$$L_k^t \le \frac{\lambda_{r_k}}{18r},\tag{75c}$$

$$L_{k}^{t} - 6\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{k}+1} - 4\left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\| \le \frac{1}{e^{t}}\left(L_{k}^{0} - 6\sqrt{\frac{\overline{\mu}r}{n_{1}}}\overline{\lambda}_{r_{k}+1} - 4\left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\right\|\right).$$
(75d)

Step 4: bounding $\|UU^{\top} - \overline{U}\overline{U}^{\top}\|$. To finish up, we still need to bound the discrepancy between U and \overline{U} . Recalling that k_{\max} satisfies $r_{k_{\max}} = r$, we can invoke (75d) and (75b) to obtain

$$\begin{split} L_{k_{\max}}^{t_{k_{\max}}} &\leq 4 \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right) \right\| + e^{-t_{k_{\max}}} \left(L_{k_{\max}}^{0} - 6\sqrt{\frac{\overline{\mu}r}{n_{1}}} \overline{\lambda}_{r_{k_{\max}}+1} - 4 \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right) \right\| \right) \\ &\leq 4 \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right) \right\| + e^{-t_{k_{\max}}} \left(7\sqrt{\frac{\overline{\mu}r}{n_{1}}} \overline{\lambda}_{r_{k_{\max}-1}+1} + \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right) \right\| \right) \\ &\leq 5 \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{Z}\right) \right\| + 7e^{-t_{k_{\max}}} \overline{\lambda}_{r_{k_{\max}-1}+1}. \end{split}$$

The $\sin\Theta$ Theorem (cf. Chen et al., 2021b, (2.26a)) then leads to

$$\begin{aligned} \left\| \boldsymbol{U}\boldsymbol{U}^{\top} - \overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}^{\top} \right\| &= \left\| \boldsymbol{U}_{k_{\max}}^{t_{k_{\max}}} \boldsymbol{U}_{k_{\max}}^{t_{k_{\max}}}^{-} - \overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}^{\top} \right\| \leq \frac{2 \|\boldsymbol{G}_{k_{\max}}^{t_{k_{\max}}} - \overline{\boldsymbol{M}}\|}{\overline{\lambda}_{r}} \\ &= \frac{2L_{k_{\max}}^{t_{k_{\max}}}}{\overline{\lambda}_{r}} \lesssim \frac{\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{Z}\right)\|}{\overline{\lambda}_{r}} + e^{-t_{k_{\max}}} \frac{\overline{\lambda}_{r_{k_{\max}-1}+1}}{\overline{\lambda}_{r}}. \end{aligned}$$
(76)

In addition, the definition of k_{max} and (75a) together show that

$$\frac{\sigma_{r_{k_{\max}-1}+1}\left(\boldsymbol{G}_{k_{\max}-1}\right)}{\sigma_{r}\left(\boldsymbol{G}_{k_{\max}-1}\right)} \leq 4.$$
(77)

In view of (75b) and Weyl's inequality, one has

$$\max_{i} \left| \sigma_{i} \left(\boldsymbol{G}_{k_{\max}-1} \right) - \overline{\lambda}_{i} \right| \leq L_{k_{\max}}^{0} = \left\| \boldsymbol{G}_{k_{\max}-1} - \overline{\boldsymbol{M}} \right\| \leq 7 \sqrt{\frac{\overline{\mu}r}{n_{1}}} \overline{\lambda}_{r_{k_{\max}-1}+1} + 5 \left\| \mathcal{P}_{\mathsf{off}-\mathsf{diag}} \left(\boldsymbol{Z} \right) \right\| \leq \frac{1}{10} \overline{\lambda}_{r_{k_{\max}-1}+1},$$

where the last inequality results from (46a) and (46b). Combine the preceding two bounds to reach

$$\overline{\lambda}_{r_{k_{\max}-1}+1} \asymp \overline{\lambda}_r. \tag{78}$$

Putting (76) together with (78) finishes the proof of Theorem 4.

B Proof of Theorem 2 ($\ell_{2,\infty}$ analysis for Deflated-HeteroPCA)

In this section, we present the proof of Theorem 2 that concerns $\ell_{2,\infty}$ statistical guarantees. For convenience, we shall continue to use the notation defined in (43a)-(43c), and again denote the SVD of $U^*\Sigma^* + EV^*$ by

$$\widetilde{U}\widetilde{\Sigma}\widetilde{W}^{\top} = U^{\star}\Sigma^{\star} + EV^{\star}, \tag{79a}$$

where $\widetilde{U} \in \mathcal{O}^{n_1,r}$, $\widetilde{\Sigma} = \mathsf{diag}(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_r)$, and $\widetilde{W} \in \mathcal{O}^{r,r}$. We can then define

$$\widetilde{\boldsymbol{M}} = \widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{\Sigma}}^{2}\widetilde{\boldsymbol{U}}^{\top} = \left(\boldsymbol{U}^{\star}\boldsymbol{\Sigma}^{\star} + \boldsymbol{E}\boldsymbol{V}^{\star}\right)\left(\boldsymbol{U}^{\star}\boldsymbol{\Sigma}^{\star} + \boldsymbol{E}\boldsymbol{V}^{\star}\right)^{\top}.$$
(79b)

In addition, we introduce

$$M^{\text{oracle}} = \widetilde{U}\widetilde{\Sigma}^{2}\widetilde{U}^{\top} + \underbrace{\mathcal{P}_{\text{off-diag}}\left(EE^{\top} - EV^{\star}V^{\star\top}E^{\top}\right)}_{=:Z},$$
(79c)

and let $U^{\text{oracle}} \in \mathcal{O}^{n_1,r}$ represent the rank-*r* leading eigen-subspace of M^{oracle} . It is easily seen that

$$\mathcal{P}_{\mathsf{off}\mathsf{-diag}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) = \mathcal{P}_{\mathsf{off}\mathsf{-diag}}\left(\boldsymbol{Y}\boldsymbol{Y}^{\top}\right) = \mathcal{P}_{\mathsf{off}\mathsf{-diag}}\left(\boldsymbol{G}_{0}\right) \quad \text{and} \quad \mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) = \mathcal{P}_{\mathsf{diag}}\left(\widetilde{\boldsymbol{M}}\right). \tag{79d}$$

Throughout this proof, we denote by $U_k^{\text{oracle}} \in \mathbb{R}^{n_1 \times r_k}$ the top- r_k eigenspace of M^{oracle} .

B.1 Several key results: eigenspace/eigenvalue perturbation and tail bounds

Before embarking on the proof of Theorem 2, we single out a couple of key results that play a crucial role in the proof. Let us begin by making note of a lemma that connects the eigenspace perturbation with a collection of polynomials of the perturbation matrix, originally developed by Xia (2021).

Lemma 1 (Xia (2021), Theorem 1). Suppose that $M = \overline{M} + Z \in \mathbb{R}^{n \times n}$, where \overline{M} and Z are both symmetric matrices. Assume that \overline{M} is rank-r with eigenvalues $\overline{\lambda}_1 \geq \cdots \geq \overline{\lambda}_r > 0$, and $\overline{U} = [\overline{u}_1, \ldots, \overline{u}_r]$ (resp. U) represents the rank-r leading eigen-subspace of \overline{M} (resp. M). If $\overline{\lambda}_r > 2 ||Z||$, then

$$\overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}^{\top} - \boldsymbol{U}\boldsymbol{U}^{\top} = \sum_{k\geq 1} \sum_{\substack{j=[j_1,\cdots,j_{k+1}]\geq \mathbf{0}\\j_1+\cdots+j_{k+1}=k}} (-1)^{\tau(j)+1} \mathfrak{P}^{-j_1} \boldsymbol{Z} \mathfrak{P}^{-j_2} \boldsymbol{Z} \cdots \boldsymbol{Z} \mathfrak{P}^{-j_{k+1}}.$$
(80)

Here, we define, for any $k \geq 1$,

$$\tau(\mathbf{j}) := \sum_{i=1}^{k+1} \mathbb{1}\{j_i > 0\},\tag{81a}$$

$$\overline{\mathbf{\Lambda}} := \operatorname{diag}\left(\bar{\lambda}_1, \dots, \bar{\lambda}_r\right),\tag{81b}$$

$$\mathfrak{P}^{0} := \overline{U}_{\perp} \overline{U}_{\perp}^{\top} = I - \overline{U} \overline{U}^{\top}, \qquad (81c)$$

$$\mathfrak{P}^{-k} := \overline{U} \overline{\Lambda}^{-k} \overline{U}^{\top}. \tag{81d}$$

As a consequence, we have

$$\left\|\overline{U}\,\overline{U}^{\top} - UU^{\top}\right\|_{2,\infty} \leq \sum_{k\geq 1} \sum_{\substack{j=[j_1,\cdots,j_{k+1}]\geq \mathbf{0}\\j_1+\cdots+j_{k+1}=k}} \left\|\mathfrak{P}^{-j_1}Z\mathfrak{P}^{-j_2}Z\cdots Z\mathfrak{P}^{-j_{k+1}}\right\|_{2,\infty}.$$
(82)

Moreover, given that we are considering multiple eigen-subspaces (e.g., $U^{\text{oracle}}, \tilde{U}, U^*$), we isolate the following result that unveils the proximity of U^{oracle} and U^* (or \tilde{U}). The proof of this result is deferred to Section B.3.

Theorem 5. Suppose that Assumption 2 holds and

$$\frac{\sigma_r^{\star}}{\omega_{\max}} \ge C_0 r \left[(n_1 n_2)^{1/4} + n_1^{1/2} \right] \log n \tag{83a}$$

$$\mu \le c_0 \frac{n_1}{r^3} \tag{83b}$$

for some large (resp. small) numerical constant $C_0 > 0$ (resp. $c_0 > 0$). Then with probability exceeding $1 - O(n^{-10})$, one has

$$\left\| \widetilde{\boldsymbol{U}} \widetilde{\boldsymbol{U}}^{\top} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \right\|_{2,\infty} \lesssim \sqrt{\frac{\mu r}{n_1}} \frac{\sqrt{n_1} \,\omega_{\mathsf{max}} \log n}{\sigma_r^{\star}},\tag{84a}$$

$$\left\| \boldsymbol{U}^{\mathsf{oracle}\top} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star\top} \right\|_{2,\infty} \lesssim \sqrt{\frac{\mu r}{n_1}} \left(\frac{\sqrt{n_1 n_2} \,\omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star^2}} + \frac{\sqrt{n_1} \,\omega_{\mathsf{max}} \log n}{\sigma_r^{\star}} \right), \tag{84b}$$

$$\left\| \boldsymbol{U}^{\mathsf{oracle}} \boldsymbol{U}^{\mathsf{oracle}\top} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star\top} \right\| \lesssim \frac{\sqrt{n_1 n_2} \,\omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star^2}} + \frac{\sqrt{n_1} \,\omega_{\mathsf{max}} \log n}{\sigma_r^{\star}}.$$
(84c)

The next two lemmas develop high-probability tail bounds on the $\ell_{2,\infty}$ norm of certain polynomials of noise matrix (with proper diagonal deletion), which are critical when invoking, say, the decomposition in Lemma 1. The proofs of these two lemmas are postponed to Sections B.4 and B.5, respectively.

Lemma 2. Suppose that Assumption 2 holds. Then with probability exceeding $1 - O(n^{-10})$, one has

$$\left\| \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\mathsf{T}}\right) \right]^{k} \boldsymbol{E}\boldsymbol{V}^{\star} \right\|_{2,\infty} \leq C_{3}\sqrt{\mu r} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{k} \omega_{\mathsf{max}} \log n$$
(85)

for all $1 \le k \le \log n$. Here, $C_3 > 0$ is some large enough numerical constant.

Lemma 3. Suppose that Assumption 2 holds. Then with probability exceeding $1 - O(n^{-10})$, one has

$$\left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top}\right) \right]^{k} \boldsymbol{U}^{\star} \right\|_{2,\infty} \leq C_{3} \sqrt{\frac{\mu r}{n_{1}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{k}$$
(86)

for all $1 \le k \le \log n$. Here, $C_3 > 0$ is some large enough numerical constant.

Finally, recall that the eigenspace perturbation theory depends heavily on both the spectral gap and the size of the perturbation matrix, which we shall study in the following lemma. In addition to these two properties, this lemma also provides an upper bound concerning the incoherence of \tilde{U} .

Lemma 4. Instate the assumptions in Theorem 5. Let us overload the notation here by setting $\sigma_{r+1}^{\star} = \widetilde{\sigma}_{r+1} = 0$, and define

$$\mathcal{R}' = \left\{ r' : 1 \le r' \le r, \left(1 - \frac{1}{2r} \right) \sigma_{r'}^{\star 2} \ge \sigma_{r'+1}^{\star 2} \right\}.$$
(87)

Then with probability exceeding $1 - O(n^{-10})$, we have

$$|\tilde{\sigma}_i - \sigma_i^*| \le \|\boldsymbol{E}\boldsymbol{V}^*\| \le \sqrt{C_5}\sqrt{n_1}\omega_{\max}\log n \tag{88a}$$

$$\widetilde{\sigma}_{r'}^2 - \widetilde{\sigma}_{r'+1}^2 \ge \frac{1}{2} \left(\sigma_{r'}^{\star 2} - \sigma_{r'+1}^{\star 2} \right), \quad \forall r' \in \mathcal{R}'$$
(88b)

$$\left\| \mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} - \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top} \right) \right\| \le 3C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n \tag{88c}$$

$$\left\| \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}} \right\|_{2,\infty} \le \frac{4C_5 \sqrt{\mu r \omega_{\max} \log n}}{\sigma_r^{\star}} \le \sqrt{\frac{\mu r}{n_1}},\tag{88d}$$

$$\left\|\widetilde{\boldsymbol{U}}\right\|_{2,\infty} \le 2\sqrt{\frac{\mu r}{n_1}} \tag{88e}$$

for some large enough constant $C_5 > 0$.

The proof of this lemma can be found in Section B.6.

B.2 Main steps for proving Theorem 2

In what follows, we shall demonstrate how to prove Theorem 2 with the assistance of Theorem 5. Reusing some of the notation in the proof of Theorem 4, we define

$$D_{k}^{t} = \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{G}_{k}^{t} - \widetilde{\boldsymbol{M}} \right) \right\|, \qquad L_{k}^{t} = \left\| \boldsymbol{G}_{k}^{t} - \widetilde{\boldsymbol{M}} \right\| \qquad and \qquad \widetilde{\boldsymbol{U}}_{k} = \widetilde{\boldsymbol{U}}_{:,1:r_{k}}$$
(89)

for any $k \ge 1$ and any $0 \le t \le t_k$. We find it helpful to introduce the following event:

$$\mathcal{E} = \{(84b), (84c), (88a), (88c) \text{ and } (88e) \text{ hold}\}.$$
(90)

The results in Lemma 4 and Theorem 5 combined with the union bound give

$$\mathbb{P}\left(\mathcal{E}\right) \ge 1 - O\left(n^{-10}\right). \tag{91}$$

Throughout the remainder of this proof, we shall assume that the event \mathcal{E} occurs unless otherwise noted. A similar argument as in the proof of (62) also tells us that

$$r_{1} \in \mathcal{R}_{1} = \left\{ r' : \frac{\sigma_{1}\left(\boldsymbol{G}_{0}\right)}{\sigma_{r'}\left(\boldsymbol{G}_{0}\right)} \leq 4 \quad \text{and} \quad \sigma_{r'}\left(\boldsymbol{G}_{0}\right) - \sigma_{r'+1}\left(\boldsymbol{G}_{0}\right) \geq \frac{1}{r}\sigma_{r'}\left(\boldsymbol{G}_{0}\right) \right\}.$$

$$(92)$$

Step 1: bounding $D_1^t = \|\mathcal{P}_{\mathsf{diag}}(\mathbf{G}_1^t - \mathbf{M})\|$. We now proceed to control the quantities $\{D_1^t\}$ for the first round. More specifically, we intend to prove, by induction, the following properties:

$$D_{1}^{t} - \left(14\sqrt{\frac{\mu r}{n_{1}}} \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2}\right) \leq \frac{1}{e^{t}} \left[D_{1}^{0} - \left(14\sqrt{\frac{\mu r}{n_{1}}} \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2}\right)\right],$$
(93a)

$$\left\|\boldsymbol{U}_{1}^{t}\boldsymbol{U}_{1}^{t\top}-\boldsymbol{U}_{1}^{\mathsf{oracle}}\boldsymbol{U}_{1}^{\mathsf{oracle}}\right\| \leq 2\frac{D_{1}^{t}}{\lambda_{r_{1}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)-\lambda_{r_{1}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} \leq \frac{1}{8},\tag{93b}$$

$$\left\|\boldsymbol{U}_{1}^{t}\right\|_{2,\infty} \leq \left\|\boldsymbol{U}_{1}^{t}\boldsymbol{U}_{1}^{t\top} - \boldsymbol{U}_{1}^{\mathsf{oracle}}\boldsymbol{U}_{1}^{\mathsf{oracle}}\right\| + \left\|\boldsymbol{U}_{1}^{\mathsf{oracle}}\right\|_{2,\infty} \leq \frac{1}{4e},\tag{93c}$$

where M^{oracle} is defined in (79c) and we recall that $U_1^{\text{oracle}} \in \mathbb{R}^{n_1 \times r_1}$ is the top- r_1 eigenspace of M^{oracle} .

Step 1.1: the base case with t = 0 for (93a)-(93c). The claim (93a) holds trivially when t = 0. Also, given that the off-diagonal entries of G_k^t and G_0 are the same, taking Zhang et al. (2022, Lemma 1) together with the property (88e) yields

$$D_1^0 = \left\| \mathcal{P}_{\mathsf{diag}}(\widetilde{\boldsymbol{M}}) \right\| = \left\| \mathcal{P}_{\mathsf{diag}}(\boldsymbol{P}_{\widetilde{\boldsymbol{U}}} \widetilde{\boldsymbol{M}} \boldsymbol{P}_{\widetilde{\boldsymbol{U}}}) \right\| \le 4 \frac{\mu r}{n_1} \| \widetilde{\boldsymbol{M}} \| = 4 \frac{\mu r}{n_1} \widetilde{\sigma}_1^2.$$
(94)

This together with (88c) further gives

$$L_{1}^{0} \leq D_{1}^{0} + \left\| \mathcal{P}_{\mathsf{off-diag}} \left(\boldsymbol{G}_{1}^{0} - \widetilde{\boldsymbol{M}} \right) \right\| = D_{1}^{0} + \left\| \mathcal{P}_{\mathsf{off-diag}} \left(\boldsymbol{G}_{0} - \widetilde{\boldsymbol{M}} \right) \right\|$$
$$\leq 4 \frac{\mu r}{n_{1}} \widetilde{\sigma}_{1}^{2} + \left\| \boldsymbol{Z} \right\| \leq 4 \frac{\mu r}{n_{1}} \widetilde{\sigma}_{1}^{2} + 3C_{5} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n, \tag{95}$$

where we remind the reader that $\boldsymbol{Z} = \mathcal{P}_{\mathsf{off}-\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^\top - \boldsymbol{E} \boldsymbol{V}^\star \boldsymbol{V}^\star^\top \boldsymbol{E}^\top \right).$

Next, let us look at the spectrum of the matrices of interest. Note that

$$\left\| \boldsymbol{M}^{\mathsf{oracle}} - \widetilde{\boldsymbol{M}} \right\| = \left\| \boldsymbol{Z} \right\|$$
 and $\left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{M}^{\mathsf{oracle}} - \widetilde{\boldsymbol{M}} \right) \right\| = \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{Z} \right) \right\| = 0.$

It comes from Weyl's inequality that, for all $1 \le i \le r+1$,

$$|\sigma_i^{\star} - \widetilde{\sigma}_i| \le \|\boldsymbol{E}\boldsymbol{V}^{\star}\| \le \sqrt{C_5}\sqrt{n_1}\omega_{\max}\log n,\tag{96}$$

$$\left|\widetilde{\sigma}_{i}^{2}-\lambda_{i}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)\right| \leq \|\boldsymbol{Z}\| \leq 3C_{5}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\mathsf{max}}^{2}\log^{2}n,\tag{97}$$

where the first line relies on (88a), and the second line results from (88c). From the assumption (20a), we can further derive

$$\forall i \in [r_1], \quad \frac{9}{10}\sigma_i^* \le \tilde{\sigma}_i \le \frac{11}{10}\sigma_i^*, \quad \frac{4}{5}\sigma_i^{*2} \le \lambda_i (\boldsymbol{G}_0) \le \frac{61}{50}\sigma_i^{*2}, \quad \lambda_{r_1+1} (\boldsymbol{G}_0) \le \frac{\sigma_{r_1}^{*2}}{100}. \tag{98}$$

Furthermore, we can easily verify that

$$\max\left\{\frac{\sigma_{1}^{\star 2}}{\sigma_{r_{1}}^{\star 2}}, \frac{\widetilde{\sigma}_{1}^{2}}{\widetilde{\sigma}_{r_{1}}^{2}}\right\} \le 8 \quad \text{and} \quad \min\left\{\frac{\widetilde{\sigma}_{r_{1}}^{2} - \widetilde{\sigma}_{r_{1}+1}^{2}}{\widetilde{\sigma}_{r_{1}}^{2}}, \frac{\sigma_{r_{1}}^{\star 2} - \sigma_{r_{1}+1}^{\star 2}}{\sigma_{r_{1}}^{\star 2}}\right\} \ge \frac{1}{2r} > 1 - \left(1 - \frac{1}{4r}\right)^{2} \tag{99}$$

and

$$\lambda_{r_1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) - \lambda_{r_1+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) \asymp \widetilde{\sigma}_{r_1}^2 - \widetilde{\sigma}_{r_1+1}^2 \asymp \sigma_{r_1}^{\star 2} - \sigma_{r_1+1}^{\star 2} \gg \|\boldsymbol{Z}\|.$$
(100)

Recall that U_1^0 (resp. U_1^{oracle}) is the top- r_1 eigenspace of G_0 (resp. M^{oracle}). With the preceding inequalities about the singular values (or eigenvalues) in place, invoking the Davis-Kahan theorem (Chen et al., 2021b, Theorem 2.7) and using (94) demonstrate that

$$\begin{aligned} \left\| \boldsymbol{U}_{1}^{0}\boldsymbol{U}_{1}^{0\top} - \boldsymbol{U}_{1}^{\mathsf{oracle}}\boldsymbol{U}_{1}^{\mathsf{oracle}\top} \right\| &\leq 2 \frac{\left\| \boldsymbol{G}_{0} - \boldsymbol{M}^{\mathsf{oracle}} \right\|}{\lambda_{r_{1}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) - \lambda_{r_{1}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} = 2 \frac{D_{1}^{0}}{\lambda_{r_{1}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) - \lambda_{r_{1}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} \\ &\lesssim \frac{\frac{\mu r}{n_{1}} \widetilde{\sigma}_{1}^{2}}{\widetilde{\sigma}_{r_{1}}^{2} - \widetilde{\sigma}_{r_{1}+1}^{2}} \lesssim \frac{\mu r^{2}}{n_{1}} \ll \sqrt{\frac{\mu r}{n_{1}}} \leq \frac{1}{16e}, \end{aligned}$$
(101)

thus validating the claim (93b) for t = 0. Here, the first inequality is valid since, according to (79d),

$$\left\| \boldsymbol{G}_{0} - \boldsymbol{M}^{\mathsf{oracle}} \right\| = \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{M}^{\mathsf{oracle}} \right) \right\| = D_{1}^{0}.$$

Moreover, in view of Theorem 5 and (88e), we can derive

$$\left\|\boldsymbol{U}^{\mathsf{oracle}}\right\|_{2,\infty} = \left\|\boldsymbol{U}^{\mathsf{oracle}}\boldsymbol{U}^{\mathsf{oracle}\top}\right\|_{2,\infty} \le \left\|\boldsymbol{U}^{\mathsf{oracle}}\boldsymbol{U}^{\mathsf{oracle}\top} - \widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{U}}^{\top}\right\|_{2,\infty} + \left\|\widetilde{\boldsymbol{U}}\right\|_{2,\infty} \le 3\sqrt{\frac{\mu r}{n_1}},\tag{102}$$

where we have also made use of the assumption (20a). Putting (101) and (102) together leads to

$$\left\| \boldsymbol{U}_{1}^{0} \right\|_{2,\infty} = \left\| \boldsymbol{U}_{1}^{0} \boldsymbol{U}_{1}^{0\top} \right\|_{2,\infty} \leq \left\| \boldsymbol{U}_{1}^{0} \boldsymbol{U}_{1}^{0\top} - \boldsymbol{U}_{1}^{\mathsf{oracle}} \boldsymbol{U}_{1}^{\mathsf{oracle}\top} \right\| + \left\| \boldsymbol{U}_{1}^{\mathsf{oracle}} \right\|_{2,\infty} \leq 4\sqrt{\frac{\mu r}{n_{1}}} \leq \frac{1}{4e},$$

which validates the claim (93c) when t = 0. We have thus established (93) for the base case.

Step 1.2: induction step for (93a)-(93c). We now move on to the inductive step. Suppose that the induction hypotheses (93a)-(93c) hold for t = t', and we would like to show their validity for t = t' + 1.

Recalling that the diagonal entries of $G_1^{t'+1}$ are equal to the diagonal entries of $U_1^{t'}\Lambda_1 U_1^{t'\top} = P_{U_1^{t'}}G_1^{t'}$ and $U_1^{t'}$ represents the rank-*r* leading singular subspace of

$$\boldsymbol{G}_{1}^{t'} = \boldsymbol{P}_{\widetilde{\boldsymbol{U}}_{1}}\widetilde{\boldsymbol{M}} + \left(\boldsymbol{G}_{1}^{t'} - \boldsymbol{P}_{\widetilde{\boldsymbol{U}}_{1}}\widetilde{\boldsymbol{M}}\right),$$

one can obtain

$$\begin{split} D_{1}^{t'+1} &= \left\| \mathcal{P}_{\mathsf{diag}} \big(\boldsymbol{G}_{1}^{t'+1} - \widetilde{\boldsymbol{M}} \big) \right\| = \left\| \mathcal{P}_{\mathsf{diag}} \big(\boldsymbol{P}_{\boldsymbol{U}_{1}^{t'}} \boldsymbol{G}_{1}^{t'} - \widetilde{\boldsymbol{M}} \big) \right\| \\ &\leq \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{P}_{\boldsymbol{U}_{1}^{t'}} \big(\boldsymbol{G}_{1}^{t'} - \widetilde{\boldsymbol{M}} \big) \right) \right\| + \left\| \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{P}_{\big(\boldsymbol{U}_{1}^{t'}\big)_{\perp}} \widetilde{\boldsymbol{M}} \boldsymbol{P}_{\widetilde{\boldsymbol{U}}} \right) \right\| \\ &\stackrel{(\mathrm{i})}{\leq} \left\| \boldsymbol{U}_{1}^{t'} \right\|_{2,\infty} \left\| \boldsymbol{G}_{1}^{t'} - \widetilde{\boldsymbol{M}} \right\| + \left\| \widetilde{\boldsymbol{U}} \right\|_{2,\infty} \left\| \big(\boldsymbol{U}_{1}^{t'} \big)_{\perp} \widetilde{\boldsymbol{M}} \right\| \\ &\stackrel{(\mathrm{ii})}{\leq} \left\| \boldsymbol{U}_{1}^{t'} \right\|_{2,\infty} L_{1}^{t'} + 2\sqrt{\frac{\mu r}{n_{1}}} \left(\left\| \big(\boldsymbol{U}_{1}^{t'} \big)_{\perp} \boldsymbol{P}_{\widetilde{\boldsymbol{U}}_{1}} \widetilde{\boldsymbol{M}} \right\| + \left\| \boldsymbol{P}_{\widetilde{\boldsymbol{U}}_{:,r_{1}+1:r}} \widetilde{\boldsymbol{M}} \right\| \right) \end{split}$$

$$\overset{\text{(iii)}}{\leq} \|\boldsymbol{U}_{1}^{t'}\|_{2,\infty} L_{1}^{t'} + 2\sqrt{\frac{\mu r}{n_{1}}} \left(2 \|\boldsymbol{G}_{1}^{t'} - \boldsymbol{P}_{\widetilde{\boldsymbol{U}}_{1}} \widetilde{\boldsymbol{M}}\| + \|\boldsymbol{P}_{\widetilde{\boldsymbol{U}}_{:,r_{1}+1:r}} \widetilde{\boldsymbol{M}}\|\right) \overset{\text{(iv)}}{\leq} \|\boldsymbol{U}_{1}^{t'}\|_{2,\infty} L_{1}^{t'} + 2\sqrt{\frac{\mu r}{n_{1}}} \left(2 \|\boldsymbol{G}_{1}^{t'} - \widetilde{\boldsymbol{M}}\| + 3 \|\boldsymbol{P}_{\widetilde{\boldsymbol{U}}_{:,r_{1}+1:r}} \widetilde{\boldsymbol{M}}\|\right) \leq \|\boldsymbol{U}_{1}^{t'}\|_{2,\infty} L_{1}^{t'} + 4\sqrt{\frac{\mu r}{n_{1}}} L_{1}^{t'} + 6\sqrt{\frac{\mu r}{n}} \widetilde{\sigma}_{r_{1}+1}^{2},$$
(103)

where (i) invokes Zhang et al. (2022, Lemma 1), (ii) results from (88e), (iii) is a consequence of Lemma 8, and (iv) applies the triangle inequality. Recognizing that (see (89))

$$L_1^{t'} \leq D_1^{t'} + \left\| \mathcal{P}_{\mathsf{off} ext{-}\mathsf{diag}} \left(\boldsymbol{G}_1^{t'} - \widetilde{\boldsymbol{M}} \right) \right\| = D_1^{t'} + \left\| \boldsymbol{Z} \right\|,$$

one can deduce that

$$D_{1}^{t'+1} \leq \left(\left\| \boldsymbol{U}_{1}^{t'} \right\|_{2,\infty}^{*} + 4\sqrt{\frac{\mu r}{n}} \right) D_{1}^{t'} + \left(\left\| \boldsymbol{U}_{1}^{t'} \right\|_{2,\infty}^{*} + 4\sqrt{\frac{\mu r}{n}} \right) \left\| \boldsymbol{Z} \right\| + 6\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2} \\ \stackrel{(93c)}{\leq} \left(\frac{1}{4e} + \frac{1}{4e} \right) D_{1}^{t'} + \left(\left\| \boldsymbol{U}_{1}^{t'} \boldsymbol{U}_{1}^{t'\top} - \boldsymbol{U}_{1}^{\mathsf{oracle}} \boldsymbol{U}_{1}^{\mathsf{oracle}\top} \right\| + \left\| \boldsymbol{U}_{1}^{\mathsf{oracle}} \right\|_{2,\infty}^{*} + 4\sqrt{\frac{\mu r}{n}} \right) \left\| \boldsymbol{Z} \right\| + 6\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2} \\ \stackrel{(93b)}{\leq} \frac{\mathrm{and}}{2e} D_{1}^{t'} + \left(2\frac{D_{1}^{t'}}{\lambda_{r_{1}} \left(\boldsymbol{M}^{\mathsf{oracle}} \right) - \lambda_{r_{1}+1} \left(\boldsymbol{M}^{\mathsf{oracle}} \right)} + 7\sqrt{\frac{\mu r}{n_{1}}} \right) \left\| \boldsymbol{Z} \right\| + 6\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2} \\ \stackrel{(100)}{\leq} \frac{1}{e} D_{1}^{t'} + 7\sqrt{\frac{\mu r}{n_{1}}} \left\| \boldsymbol{Z} \right\| + 6\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2}.$$

$$(104)$$

This together with the induction hypotheses further leads to

$$\begin{split} D_1^{t'+1} - \left(14\sqrt{\frac{\mu r}{n_1}} \, \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_1+1}^2 \right) &\leq \frac{1}{e} \left[D_1^{t'} - \left(14\sqrt{\frac{\mu r}{n_1}} \, \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_1+1}^2 \right) \right] \\ &\leq \frac{1}{e^{t'+1}} \left[D_1^0 - \left(14\sqrt{\frac{\mu r}{n_1}} \, \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_1+1}^2 \right) \right], \end{split}$$

thus justifying the induction hypothesis (93a) for t = t' + 1.

In addition, (104) allows us to derive

$$\begin{split} \frac{D_{1}^{t'+1}}{\lambda_{r_{1}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) - \lambda_{r_{1}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} &\leq \frac{\frac{1}{e}D_{1}^{t'} + 7\sqrt{\frac{\mu r}{n_{1}}} \, \|\boldsymbol{Z}\| + 6\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2}}{\lambda_{r_{1}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) - \lambda_{r_{1}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} \\ &\leq \frac{1}{e} \cdot \frac{1}{8} + \frac{7\sqrt{\frac{\mu r}{n_{1}}} \, \|\boldsymbol{Z}\|}{\lambda_{r_{1}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) - \lambda_{r_{1}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} + \frac{C_{5}\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{1}+1}^{2}}{\widetilde{\sigma}_{r_{1}}^{2} - \widetilde{\sigma}_{r_{1}+1}^{2}} \\ &\leq \frac{1}{8e} + \frac{1}{80} + C_{5}\sqrt{\frac{\mu r^{3}}{n_{1}}} \leq \frac{1}{16}, \end{split}$$

where the second line invokes the induction hypothesis (93b) (when t = t') and (100), and the last line relies on (99) and the assumption (20b).

Recalling that $\mathcal{P}_{\mathsf{off}-\mathsf{diag}}(M^{\mathsf{oracle}}) = \mathcal{P}_{\mathsf{off}-\mathsf{diag}}(YY^{\top}) = \mathcal{P}_{\mathsf{off}-\mathsf{diag}}(G_k^t)$ and $\mathcal{P}_{\mathsf{off}-\mathsf{diag}}(M^{\mathsf{oracle}}) = \mathcal{P}_{\mathsf{off}-\mathsf{diag}}(\widetilde{M})$, one has

$$\left\|\boldsymbol{G}_{k}^{t}-\boldsymbol{M}^{\mathsf{oracle}}\right\|=\left\|\mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{G}_{k}^{t}-\boldsymbol{M}^{\mathsf{oracle}}\right)\right\|=\left\|\mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{G}_{k}^{t}-\widetilde{\boldsymbol{M}}\right)\right\|=D_{k}^{t}.$$
(105)

Therefore, we can readily apply the Davis-Kahan theorem (Chen et al., 2021b, Theorem 2.7) to arrive at

$$\left\|\boldsymbol{U}_{1}^{t'+1}\boldsymbol{U}_{1}^{t'+1\top}-\boldsymbol{U}_{1}^{\mathsf{oracle}}\boldsymbol{U}_{1}^{\mathsf{oracle}}^{\top}\right\| \leq 2\frac{D_{1}^{t'+1}}{\lambda_{r_{1}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)-\lambda_{r_{1}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} \leq \frac{1}{8}.$$

Here, we remind the readers that U_k^t (resp. U_k^{oracle}) represents the top- r_k eigenspace of G_k^t (resp. M^{oracle}). This establishes the induction hypothesis (93b) for t = t' + 1, which in turn also validates (93c) for t = t' + 1.

Therefore, we have finished the proof for the hypotheses (93a)-(93c) when t = t' + 1, thereby completing the induction step for the first round.

Step 2: bounding $D_k^t = \|\mathcal{P}_{\mathsf{diag}}(\mathbf{G}_k^t - \widetilde{\mathbf{M}})\|$ for k > 1. Having established the desired properties for the first round, we would like extend these to accommodate $\{D_k^t\}$ for the k-th round with k > 1. More precisely, we would like to further bound $\{\|\mathcal{P}_{\mathsf{diag}}(\mathbf{G}_k^t - \widetilde{\mathbf{M}})\|\}_{k>1,t\geq 0}$ by means of a recursive argument. To begin with, in view of (88c) and (99), by choosing

$$t_1 \ge \log\left(C\frac{\sigma_1^{\star 2}}{\sigma_{r_1+1}^{\star 2}}\right) \ge \log\left(\frac{C\sqrt{\frac{\mu r}{n_1}}\sigma_1^{\star 2}}{3C_5\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log^2 n + \sigma_{r_1+1}^{\star 2}}\right),$$

we have

$$\begin{split} D_2^0 &= D_1^{t_1} \le 14\sqrt{\frac{\mu r}{n_1}} \, \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_1+1}^2 + 3C_5\sqrt{\frac{\mu r}{n_1}} \left(\sqrt{n_1 n_2} + n_1\right) \omega_{\max}^2 \log^2 n + \sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_1+1}^2 \\ &\le 45C_5\sqrt{\frac{\mu r}{n_1}} \left(\sqrt{n_1 n_2} + n_1\right) \omega_{\max}^2 \log^2 n + 13\sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_1+1}^2. \end{split}$$

Repeating similar arguments as in (99) and (100) yields

$$\max\left\{\frac{\sigma_{r_1+1}^{\star 2}}{\sigma_{r_2}^{\star 2}}, \ \frac{\widetilde{\sigma}_{r_1+1}^2}{\widetilde{\sigma}_{r_2}^2}\right\} \le 8 \quad \text{and} \quad \min\left\{\frac{\widetilde{\sigma}_{r_2}^2 - \widetilde{\sigma}_{r_2+1}^2}{\widetilde{\sigma}_{r_2}^2}, \ \frac{\sigma_{r_2}^{\star 2} - \sigma_{r_2+1}^{\star 2}}{\sigma_{r_2}^{\star 2}}\right\} \ge \frac{1}{2r} > 1 - \left(1 - \frac{1}{4r}\right)^2 \tag{106}$$

and

$$\lambda_{r_2} \left(\boldsymbol{M}^{\mathsf{oracle}} \right) - \lambda_{r_2+1} \left(\boldsymbol{M}^{\mathsf{oracle}} \right) \asymp \widetilde{\sigma}_{r_2}^2 - \widetilde{\sigma}_{r_2+1}^2 \asymp \sigma_{r_2}^{\star 2} - \sigma_{r_2+1}^{\star 2} \gg \| \boldsymbol{Z} \|.$$
(107)

We can then reach

$$\begin{split} \frac{D_2^0}{\lambda_{r_2} \left(\boldsymbol{M}^{\text{oracle}} \right) - \lambda_{r_2+1} \left(\boldsymbol{M}^{\text{oracle}} \right)} \lesssim \frac{45C_5 \sqrt{\frac{\mu r}{n_1}} \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n}{\sigma_{r_2}^* - \sigma_{r_2+1}^*} + \frac{13 \sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_1+1}^2}{\widetilde{\sigma}_{r_2}^2 - \widetilde{\sigma}_{r_2+1}^2} \\ \lesssim \sqrt{\frac{\mu r}{n_1}} + \sqrt{\frac{\mu r}{n_1}} \frac{\widetilde{\sigma}_{r_1+1}^2}{\widetilde{\sigma}_{r_2}^2} \cdot \frac{\widetilde{\sigma}_{r_2}^2}{\widetilde{\sigma}_{r_2}^2 - \widetilde{\sigma}_{r_2+1}^2} \\ \lesssim \sqrt{\frac{\mu r^3}{n_1}} \ll \frac{1}{8\sqrt{2}}. \end{split}$$

Thus, invoking the Davis-Kahan theorem (Chen et al., 2021b, Theorem 2.7) and (105) leads to

$$\left\|\boldsymbol{U}_{2}^{0}\boldsymbol{U}_{2}^{0\top}-\boldsymbol{U}_{2}^{\mathsf{oracle}}\boldsymbol{U}_{2}^{\mathsf{oracle}\top}\right\| \leq \sqrt{2}\frac{D_{2}^{0}}{\lambda_{r_{2}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)-\lambda_{r_{2}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} \lesssim \sqrt{\frac{\mu r^{3}}{n_{1}}} \ll \frac{1}{8},$$

where we recall that U_k^t (resp. U_k^{oracle}) is the top- r_k eigenspace of G_k^t (resp. M^{oracle}). Similar to the argument for (103), one can obtain

$$D_2^{t+1} \le \left\| \boldsymbol{U}_2^t \right\|_{2,\infty} L_2^t + 4\sqrt{\frac{\mu r}{n_1}} L_2^t + 6\sqrt{\frac{\mu r}{n_1}} \widetilde{\sigma}_{r_2+1}^2.$$
(108)

Further, repeat similar arguments as in (75a), (93a)-(93c), (99), (100) and (108) to yield that: for all $1 \le k \le k_{\max}$ and $1 \le t \le t_k$, one has the following properties:

$$r_k \in \mathcal{R}_k$$
 where \mathcal{R}_k is defined in (75a), (109a)

$$D_{k}^{t} - \left(14\sqrt{\frac{\mu r}{n_{1}}} \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{k}+1}^{2}\right) \leq \frac{1}{e^{t}} \left[D_{k}^{0} - \left(14\sqrt{\frac{\mu r}{n_{1}}} \|\boldsymbol{Z}\| + 12\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{k}+1}^{2}\right)\right],$$
(109b)

$$\left\|\boldsymbol{U}_{k}^{t}\boldsymbol{U}_{k}^{t\top}-\boldsymbol{U}_{k}^{\mathsf{oracle}}\boldsymbol{U}_{k}^{\mathsf{oracle}}^{\top}\right\| \leq 2\frac{D_{k}^{t}}{\lambda_{r_{k}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)-\lambda_{r_{k}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right)} \leq \frac{1}{8},\tag{109c}$$

$$\left\|\boldsymbol{U}_{k}^{t}\right\|_{2,\infty} \leq \left\|\boldsymbol{U}_{k}^{t}\boldsymbol{U}_{k}^{t\top} - \boldsymbol{U}_{k}^{\mathsf{oracle}}\boldsymbol{U}_{k}^{\mathsf{oracle}\top}\right\| + \left\|\boldsymbol{U}_{k}^{\mathsf{oracle}}\right\|_{2,\infty} \leq \frac{1}{4},\tag{109d}$$

$$D_{k+1}^{0} = D_{k}^{t_{k}} \le 45C_{5}\sqrt{\frac{\mu r}{n_{1}}}\left(\sqrt{n_{1}n_{2}} + n_{1}\right)\omega_{\max}^{2}\log^{2}n + 13\sqrt{\frac{\mu r}{n_{1}}}\widetilde{\sigma}_{r_{k}+1}^{2}.$$
 (109e)

$$\max\left\{\frac{\sigma_{r_k+1}^{\star 2}}{\sigma_{r_k}^{\star 2}}, \ \frac{\widetilde{\sigma}_{r_k+1}^2}{\widetilde{\sigma}_{r_k}^2}\right\} \le 8 \quad \text{and} \quad \min\left\{\frac{\widetilde{\sigma}_{r_k}^2 - \widetilde{\sigma}_{r_k+1}^2}{\widetilde{\sigma}_{r_k}^2}, \ \frac{\sigma_{r_k}^{\star 2} - \sigma_{r_k+1}^{\star 2}}{\sigma_{r_k}^{\star 2}}\right\} \ge \frac{1}{2r},\tag{109f}$$

$$\lambda_{r_{k}}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) - \lambda_{r_{k}+1}\left(\boldsymbol{M}^{\mathsf{oracle}}\right) \asymp \widetilde{\sigma}_{r_{k}}^{2} - \widetilde{\sigma}_{r_{k}+1}^{2} \asymp \sigma_{r_{k}}^{\star 2} - \sigma_{r_{k}+1}^{\star 2} \gg \left(\sqrt{n_{1}n_{2}} + n_{1}\right)\omega_{\mathsf{max}}^{2}\log^{2}n, \quad (109g)$$

$$D_{k}^{t+1} \leq \left\| \boldsymbol{U}_{k}^{t} \right\|_{2,\infty} L_{k}^{t} + 4\sqrt{\frac{\mu r}{n_{1}}} L_{k}^{t} + 6\sqrt{\frac{\mu r}{n_{1}}} \widetilde{\sigma}_{r_{k}+1}^{2}, \tag{109h}$$

provided that the numbers of iterations t_i satisfy (16a)-(16b). Here, we remind the reader that U_k^{oracle} represents the top- r_k eigenspace of M^{oracle} . Given that these can be established using exactly the same arguments as before, we omit the details here for the sake of brevity.

By letting $k = k_{max}$ in (109b) and (109e) and recalling (97), we immediately have

$$D_{k_{\max}}^{t_{k_{\max}}} \lesssim \sqrt{\frac{\mu r}{n_1}} \left(\sqrt{n_1 n_2} + n_1\right) \omega_{\max}^2 \log^2 n.$$

Then the Davis-Kahan $\sin\Theta$ theorem reveals that

$$\|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\text{oracle}}\boldsymbol{U}^{\text{oracle}^{\top}}\| \lesssim \frac{\|\boldsymbol{G}_{k_{\max}}^{t_{\max}} - \boldsymbol{M}^{\text{oracle}}\|}{\lambda_{r_{k_{\max}}}(\boldsymbol{M}^{\text{oracle}}) - \lambda_{r_{k_{\max}}+1}(\boldsymbol{M}^{\text{oracle}})} = \frac{D_{k_{\max}}^{t_{k_{\max}}}}{\lambda_{r_{k_{\max}}}(\boldsymbol{M}^{\text{oracle}}) - \lambda_{r_{k_{\max}}+1}(\boldsymbol{M}^{\text{oracle}})}$$
$$\lesssim \frac{\sqrt{\frac{\mu r}{n_{1}}}\left(\sqrt{n_{1}n_{2}} + n_{1}\right)\omega_{\max}^{2}\log^{2}n}}{\sigma_{r_{k_{\max}}}^{\star 2} - \sigma_{r_{k_{\max}}+1}^{\star 2}}}$$
$$\lesssim \frac{\sqrt{\frac{\mu r}{n_{1}}}\left(\sqrt{n_{1}n_{2}} + n_{1}\right)\omega_{\max}^{2}\log^{2}n}}{\sigma_{r_{k_{\max}}}^{\star 2} - \sigma_{r_{k_{\max}}+1}^{\star 2}}}$$
$$\approx \frac{\sqrt{\frac{\mu r}{n_{1}}}\left(\sqrt{n_{1}n_{2}} + n_{1}\right)\omega_{\max}^{2}\log^{2}n}}{\sigma_{r_{k}}^{\star 2}}, \tag{110}$$

where the first line also applies (79d), the third line relies on (109g), and the last line holds since $r_{k_{max}} = r$.

Step 3: bounding $||UR_U - U^*||_{2,\infty}$ and $||UR_U - U^*||$. In the final step, we invoke Theorem 5 to establish the desired bounds on $||UR_U - U^*||$ and $||UR_U - U^*||_{2,\infty}$. To begin with, inequality (110) taken together with Theorem 5 gives

$$\|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\|_{2,\infty} \leq \|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\text{oracle}}\boldsymbol{U}^{\text{oracle}\top}\| + \|\boldsymbol{U}^{\text{oracle}}\boldsymbol{U}^{\text{oracle}\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\|_{2,\infty}$$
$$\leq \sqrt{\frac{\mu r}{n_1}} \left(\frac{\sqrt{n_1 n_2}\omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}} + \frac{\sqrt{n_1}\omega_{\max} \log n}{\sigma_r^{\star}}\right)$$
(111)

and

$$\begin{split} \left\| \boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top} \right\| &\leq \left\| \boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\mathsf{oracle}}\boldsymbol{U}^{\mathsf{oracle}\top} \right\| + \left\| \boldsymbol{U}^{\mathsf{oracle}}\boldsymbol{U}^{\mathsf{oracle}\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top} \right\| \\ &\lesssim \frac{\sqrt{\frac{\mu r}{n_1}} \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star^2}} + \frac{\sqrt{n_1 n_2} \omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star^2}} + \frac{\sqrt{n_1} \omega_{\mathsf{max}} \log n}{\sigma_r^{\star}} \end{split}$$

$$\lesssim \frac{\sqrt{n_1 n_2} \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}} + \frac{\sqrt{n_1} \omega_{\max} \log n}{\sigma_r^{\star}}.$$
(112)

As an immediate consequence of (111) and Definition 1, we have

$$\|\boldsymbol{U}\|_{2,\infty} = \|\boldsymbol{U}\boldsymbol{U}^{\top}\|_{2,\infty} \le \|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\|_{2,\infty} + \|\boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\|_{2,\infty} \le \frac{\sqrt{\frac{\mu r}{n_1}} \left(\sqrt{n_1 n_2} + n_1\right) \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}} + \sqrt{\frac{\mu r}{n_1}} \le 2\sqrt{\frac{\mu r}{n_1}}.$$
(113)

Recalling that $\mathbf{R}_{U} = \operatorname{sgn}(U^{\top}U^{\star})$, one can invoke Chen et al. (2021b, Eqn. (4.123) and Lemma 2.5) to obtain

$$\left\|\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\top}\boldsymbol{U}^{\star}\right\| \leq \left\|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\right\|^{2}.$$
(114)

We can then arrive at

$$\begin{split} \|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\|_{2,\infty} &\leq \|\boldsymbol{U}\left(\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\top}\boldsymbol{U}^{\star}\right)\|_{2,\infty} + \|\boldsymbol{U}\boldsymbol{U}^{\top}\boldsymbol{U}^{\star} - \boldsymbol{U}^{\star}\|_{2,\infty} \\ &\leq \|\boldsymbol{U}\|_{2,\infty} \left\|\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\top}\boldsymbol{U}^{\star}\right\| + \left\|\left(\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star^{\top}}\right)\boldsymbol{U}^{\star}\right\|_{2,\infty} \\ &\leq \|\boldsymbol{U}\|_{2,\infty} \left\|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star^{\top}}\right\|^{2} + \left\|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star^{\top}}\right\|_{2,\infty} \|\boldsymbol{U}^{\star}\| \\ &\lesssim \sqrt{\frac{\mu r}{n_{1}}} \left(\frac{\sqrt{n_{1}n_{2}}\omega_{\max}^{2}\log^{2}n}{\sigma_{r}^{\star^{2}}} + \frac{\sqrt{n_{1}}\omega_{\max}\log n}{\sigma_{r}^{\star}}\right)^{2} \\ &+ \sqrt{\frac{\mu r}{n_{1}}} \left(\frac{\sqrt{n_{1}n_{2}}\omega_{\max}^{2}\log^{2}n}{\sigma_{r}^{\star^{2}}} + \frac{\sqrt{n_{1}}\omega_{\max}\log n}{\sigma_{r}^{\star}}\right) \\ &\asymp \sqrt{\frac{\mu r}{n_{1}}} \left(\frac{\sqrt{n_{1}n_{2}}\omega_{\max}^{2}\log^{2}n}{\sigma_{r}^{\star^{2}}} + \frac{\sqrt{n_{1}}\omega_{\max}\log n}{\sigma_{r}^{\star}}\right), \end{split}$$

where the third line makes use of (114), the fourth line invokes (111), (112) and (113), and the last line results from the assumption (20a). In addition, inequality (112) and $\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \leq \sqrt{2} \|\boldsymbol{U}\boldsymbol{U}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\|$ (see the proof of Chen et al. (2021b, Lemma 2.6)) taken collectively yield

$$\| \boldsymbol{U} \boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star} \| \lesssim \| \boldsymbol{U} \boldsymbol{U}^{ op} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star op} \| \lesssim rac{\sqrt{n_1 n_2} \omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star 2}} + rac{\sqrt{n_1} \omega_{\mathsf{max}} \log n}{\sigma_r^{\star}}.$$

This concludes the proof.

B.3 Proof of Theorem 5

Let us define the following event:

$$\mathcal{E}' := \{(85) \text{ and } (86) \text{ hold for } 0 \le k \le \log n\} \cap \{(88a), (88b), (88c), (88d) \text{ and } (88e) \text{ hold}\}.$$
 (115)

Then Lemma 2, Lemma 3, Lemma 4 and the union bound taken collectively imply that

$$\mathbb{P}\left(\mathcal{E}'\right) \ge 1 - O\left(n^{-10}\right). \tag{116}$$

In the rest of the proof, we shall assume that \mathcal{E}' occurs unless otherwise noted.

Recall that $\mathbf{Z} = \mathcal{P}_{\text{off-diag}} \left(\mathbf{E} \mathbf{E}^{\top} - \mathbf{E} \mathbf{V}^{\star} \mathbf{V}^{\star \top} \mathbf{E}^{\top} \right)$ (see (79c)) and that $\widetilde{\mathbf{U}} \widetilde{\mathbf{\Sigma}} \widetilde{\mathbf{W}}^{\top}$ denotes the SVD of $\mathbf{U}^{\star} \mathbf{\Sigma}^{\star} + \mathbf{E} \mathbf{V}^{\star} \in \mathbb{R}^{n_1 \times r}$ (cf. (79a)). In view of Lemma 1, to bound $\| \mathbf{U}^{\text{oracle}} \mathbf{U}^{\text{oracle}} - \widetilde{\mathbf{U}} \widetilde{\mathbf{U}}^{\top} \|_{2,\infty}$, it suffices to (i) bound each of the terms $\| \mathfrak{P}^{-j_1} \mathbf{Z} \mathfrak{P}^{-j_2} \mathbf{Z} \cdots \mathbf{Z} \mathfrak{P}^{-j_{k+1}} \|_{2,\infty}$ for $1 \leq k \leq \log n$, where $\mathbf{j} = [j_1, \cdots, j_{k+1}] \geq \mathbf{0}$ and $j_1 + \cdots + j_{k+1} = k$; and (ii) show that the total contribution of the remaining terms on the right-hand side of (82) is well-controlled. Based on these ideas, our proof consists of four steps below.

Step 1: bounding $\|Z^i \widetilde{U}\|_{2,\infty}$. We start by bounding a simpler term $\|Z^i \widetilde{U}\|_{2,\infty}$. It follows from (88a) that

$$\left\|\widetilde{\boldsymbol{\Sigma}}^{-1}\right\| \le \frac{1}{\sigma_r^{\star} - \left\|\boldsymbol{E}\boldsymbol{V}^{\star}\right\|} \le \frac{\sqrt{2}}{\sigma_r^{\star}}.$$
(117)

It is also observed from (79a) that

$$\widetilde{U} = (U^{\star} \Sigma^{\star} + EV^{\star}) \widetilde{W} \widetilde{\Sigma}^{-1}$$
(118)

$$= U^{\star}U^{\star\top} (U^{\star}\Sigma^{\star} + EV^{\star}) \widetilde{W}\widetilde{\Sigma}^{-1} + (EV^{\star} - U^{\star}U^{\star\top}EV^{\star}) \widetilde{W}\widetilde{\Sigma}^{-1}$$

$$= U^{\star}U^{\star\top}\widetilde{U} + (EV^{\star} - U^{\star}U^{\star\top}EV^{\star}) \widetilde{W}\widetilde{\Sigma}^{-1}.$$
 (119)

As a consequence, $oldsymbol{Z}^i \widetilde{oldsymbol{U}}$ admits the following decomposition:

$$\begin{aligned} \boldsymbol{Z}^{i} \widetilde{\boldsymbol{U}} &= \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star \top} \boldsymbol{E}^{\top} \right) \right]^{i} \widetilde{\boldsymbol{U}} \\ &= -\sum_{j=0}^{i-1} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{j} \mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star \top} \boldsymbol{E}^{\top} \right) \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star \top} \boldsymbol{E}^{\top} \right) \right]^{i-j-1} \widetilde{\boldsymbol{U}} \\ &+ \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \widetilde{\boldsymbol{U}} \\ &= -\sum_{j=0}^{i-1} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{j} \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star \top} \boldsymbol{E}^{\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star \top} \boldsymbol{E}^{\top} \right) \right]^{i-j-1} \widetilde{\boldsymbol{U}} \\ &+ \sum_{j=0}^{i-1} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{j} \mathcal{P}_{\text{diag}} \left(\boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star \top} \boldsymbol{E}^{\top} \right) \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star \top} \boldsymbol{E}^{\top} \right) \right]^{i-j-1} \widetilde{\boldsymbol{U}} \\ &+ \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \widetilde{\boldsymbol{U}} \\ &+ \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \left(\boldsymbol{E} \boldsymbol{V}^{\star} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right) \widetilde{\boldsymbol{W}} \widetilde{\boldsymbol{\Sigma}}^{-1}, \end{aligned}$$
(120)

where the second identity is valid due to the following relation

$$(A + B)^{i} = B^{i} + \sum_{j=0}^{i-1} B^{j} A (A + B)^{i-j-1}$$

that holds for any matrices $A, B \in \mathbb{R}^{n_1 \times n_1}$, and the third identity in (120) arises from (119). This allows us to bound $\|Z^i \widetilde{U}\|_{2,\infty}$, for any $1 \le i \le \log n$, as follows:

$$\begin{split} &|\boldsymbol{Z}^{i} \widetilde{\boldsymbol{U}}\|_{2,\infty} \\ &\leq \sum_{j=0}^{i-1} \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{j} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{V}^{\star\top} \boldsymbol{E}^{\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star\top} \boldsymbol{E}^{\top} \right) \right]^{i-j-1} \widetilde{\boldsymbol{U}} \right\| \\ &+ \sum_{j=0}^{i-1} \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{j} \mathcal{P}_{\text{diag}} \left(\boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star\top} \boldsymbol{E}^{\top} \right) \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star\top} \boldsymbol{E}^{\top} \right) \right]^{i-j-1} \widetilde{\boldsymbol{U}} \right\| \\ &+ \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \boldsymbol{U}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{U}^{\star\top} \widetilde{\boldsymbol{U}} \right\| + \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{U}^{\star\top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \left\| \widetilde{\boldsymbol{W}} \right\| \left\| \widetilde{\boldsymbol{\Sigma}^{-1}} \right\| \\ &+ \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \boldsymbol{U}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{U}^{\star\top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \left\| \widetilde{\boldsymbol{W}} \right\| \left\| \widetilde{\boldsymbol{\Sigma}^{-1}} \right\| \\ &\leq \sum_{j=0}^{i-1} \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{j} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \left\| \mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star\top} \boldsymbol{E}^{\top} \right) \right\|^{i-j-1} \\ &+ \sum_{j=0}^{i-1} \left\| \mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right\|^{j} \left\| \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty}^{2} \left\| \mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \boldsymbol{E} \boldsymbol{V}^{\star} \boldsymbol{V}^{\star\top} \boldsymbol{E}^{\top} \right) \right\|^{i-j-1} \end{aligned}$$

$$+ \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \boldsymbol{U}^{\star} \right\|_{2,\infty} + \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{\tilde{\Sigma}}^{-1} \right\| \\ + \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{i} \boldsymbol{U}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \left\| \boldsymbol{\tilde{\Sigma}}^{-1} \right\| \\ \leq \sum_{j=0}^{i-1} \left(C_{3} \sqrt{\mu r} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{j} \omega_{\max} \log n \right) \cdot C_{5} \sqrt{n_{1}} \omega_{\max} \log n \\ \cdot \left(3C_{5} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{i-j-1} \\ + \sum_{j=0}^{i-1} \left(C_{5} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{j} \left(C_{3} \sqrt{\mu r} \omega_{\max} \log n \right)^{2} \cdot \left(3C_{5} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{i-j-1} \\ + C_{3} \sqrt{\frac{\mu r}{n_{1}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{i} \omega_{\max} \log n \cdot \frac{2}{\sigma_{r}^{\star}} \\ + C_{3} \sqrt{\frac{\mu r}{n_{1}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{i} \cdot C_{5} \sqrt{n_{1}} \omega_{\max} \log n \cdot \frac{2}{\sigma_{r}^{\star}} \\ \leq 4C_{3} \sqrt{\frac{\mu r}{n_{1}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{i}, \qquad (121)$$

provided that $C_3 \ge 6C_5$ and $\sigma_r^* \ge C_0 \sqrt{n_1} \omega_{\max} \log n$. Here, the first inequality relies on (120) and the triangle inequality, the second inequality makes use of $\|\widetilde{\boldsymbol{W}}\| = 1$, whereas the third inequality results from (85), (86), (88a), (88c) and (117).

Step 2: bounding the sum for small k. For any $1 \le k \le \log n$ and any (j_1, \ldots, j_{k+1}) satisfying $j_1, \ldots, j_{k+1} \ge 0$ and $j_1 + \cdots + j_{k+1} = k$, let ℓ be the smallest i such that $j_i \ne 0$. We define the matrices

$$\widetilde{\mathfrak{P}}^{-j} = \widetilde{U}\widetilde{\Sigma}^{-2j}\widetilde{U}^{\top} \quad (j \ge 1) \qquad \text{and} \qquad \widetilde{\mathfrak{P}}^{0} = \widetilde{U}_{\perp}\widetilde{U}_{\perp}^{\top}, \tag{122}$$

where we remind the reader that $\widetilde{\Sigma} = \text{diag}(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_r)$ is the diagonal matrix containing the nonzero singular values of $U^* \Sigma^* + EV^*$. Noting that $\|\widetilde{\mathfrak{P}}^{-j}\| = \|\widetilde{\Sigma}^{-1}\|^{2j}$ and $\sum_{i=1}^{k+1} j_i = \sum_{\ell=1}^{k+1} j_i = k$ (using the definition of ℓ), one has

$$\prod_{i=\ell}^{k+1} \left\| \widetilde{\mathfrak{P}}^{-j_i} \right\| = \left\| \widetilde{\Sigma}^{-1} \right\|^{2k}.$$
(123)

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It then follows from (117), (123) and the definition of ℓ that

$$\left\|\widetilde{\mathfrak{P}}^{-j_{\ell}}\boldsymbol{Z}\cdots\boldsymbol{Z}\widetilde{\mathfrak{P}}^{-j_{k+1}}\right\| \leq \left\|\boldsymbol{Z}\right\|^{k-\ell+1}\prod_{i=\ell}^{k+1}\left\|\widetilde{\mathfrak{P}}^{-j_{i}}\right\| = \left\|\boldsymbol{Z}\right\|^{k-\ell+1}\left\|\widetilde{\boldsymbol{\Sigma}}^{-1}\right\|^{2k} \leq \left\|\boldsymbol{Z}\right\|^{k-\ell+1}\left(\frac{2}{\sigma_{r}^{\star 2}}\right)^{k}$$
(124)

and for $1 \leq i \leq \ell - 1$,

$$\left\| \boldsymbol{Z} \widetilde{\boldsymbol{\mathfrak{P}}}^{-j_{i+1}} \boldsymbol{Z} \cdots \boldsymbol{Z} \widetilde{\boldsymbol{\mathfrak{P}}}^{-j_{k+1}} \right\| \leq \left\| \boldsymbol{Z} \right\|^{k-i+1} \left\| \widetilde{\boldsymbol{\Sigma}}^{-1} \right\|^{2k} \leq \left\| \boldsymbol{Z} \right\|^{k-i+1} \left(\frac{2}{\sigma_r^{\star 2}} \right)^{\kappa}.$$
(125)

We can see from the definition of ℓ and $\widetilde{\mathfrak{P}}^0$ that

$$\widetilde{\mathfrak{P}}^{-j_1} Z \widetilde{\mathfrak{P}}^{-j_2} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}} = Z^{\ell-1} \widetilde{\mathfrak{P}}^{-j_\ell} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}} - \sum_{i=1}^{\ell-1} Z^{i-1} P_{\widetilde{U}} Z \widetilde{\mathfrak{P}}^{-j_{i+1}} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}}, \qquad (126)$$

which allows us to derive

$$\left\|\widetilde{\mathfrak{P}}^{-j_1}Z\widetilde{\mathfrak{P}}^{-j_2}Z\cdots Z\widetilde{\mathfrak{P}}^{-j_{k+1}}
ight\|_{2,\infty}$$

$$\leq \left\| Z^{\ell-1} \widetilde{\mathfrak{P}}^{-j_{\ell}} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}} \right\|_{2,\infty} + \sum_{i=1}^{\ell-1} \left\| Z^{i-1} P_{\widetilde{U}} Z \widetilde{\mathfrak{P}}^{-j_{i+1}} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}} \right\|_{2,\infty} \\
\leq \left\| Z^{\ell-1} \widetilde{U} \right\|_{2,\infty} \left\| \widetilde{\mathfrak{P}}^{-j_{\ell}} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}} \right\| + \sum_{i=1}^{\ell-1} \left\| Z^{i-1} \widetilde{U} \right\|_{2,\infty} \left\| Z \widetilde{\mathfrak{P}}^{-j_{i+1}} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}} \right\| \\
\leq \left[4C_3 \sqrt{\frac{\mu r}{n_1}} \left(C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n \right)^{\ell-1} \right] \cdot \frac{\left(3C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n \right)^{k-\ell+1}}{\left(\sigma_r^{\star 2}/2 \right)^k} \\
+ \sum_{i=1}^{\ell-1} \left[4C_3 \sqrt{\frac{\mu r}{n_1}} \left(C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n \right)^{i-1} \right] \cdot \frac{\left(3C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n \right)^{k-i+1}}{\left(\sigma_r^{\star 2}/2 \right)^k} \\
\leq 4C_3 \sqrt{\frac{\mu r}{n_1}} \left(\frac{C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}} \right)^k \cdot \sum_{i=1}^{\ell} \frac{1}{2^{k-i+1}} \\
\leq 8C_3 \sqrt{\frac{\mu r}{n_1}} \left(\frac{C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}} \right)^k.$$
(127)

Here, the first inequality comes from (126) and the triangle inequality, the second inequality holds due to the definition of ℓ , the basic inequality $\|AB\|_{2,\infty} \leq \|A\|_{2,\infty} \|B\|$ and the fact $\|\widetilde{U}^{\top}C\| \leq \|C\|$, the third inequality is a consequence of (88c), (121), (124) and (125), and the second last inequality is valid as long as $C_3 \geq 12C_5$.

Step 3: bounding the sum for large k. For any $k \ge \lfloor \log n \rfloor + 1$, the signal-to-noise condition (83a) implies that there exists a large constant C > 0 such that

$$\left(\frac{3C_5\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log^2 n}{\sigma_r^{\star 2}/2}\right)^{k-1} \le \left(\frac{1}{C^2}\right)^{k-1} \le \frac{1}{C^k}.$$

It is also seen that

$$\left| \left\{ (j_1, \dots, j_{k+1}) : j_1, \dots, j_{k+1} \ge 0 \text{ and } j_1 + \dots + j_{k+1} = k \right\} \right| = {\binom{2k+1}{k}} \le 4^k,$$

In view of (88c), (117) and (123), we have

$$\sum_{\substack{j_1,\dots,j_{k+1}\geq 0\\j_1+\dots+j_{k+1}=k}} \left\| \widetilde{\mathfrak{P}}^{-j_1} Z \widetilde{\mathfrak{P}}^{-j_2} Z \cdots Z \widetilde{\mathfrak{P}}^{-j_{k+1}} \right\|_{2,\infty} \\
\leq \sum_{\substack{j_1,\dots,j_{k+1}\geq 0\\j_1+\dots+j_{k+1}=k}} \left\| \widetilde{\mathfrak{P}}^{-j_1} \right\| \| Z \| \left\| \widetilde{\mathfrak{P}}^{-j_2} \right\| \| Z \| \cdots \| Z \| \left\| \widetilde{\mathfrak{P}}^{-j_{k+1}} \right\| \\
\leq \sum_{\substack{j_1,\dots,j_{k+1}\geq 0\\j_1+\dots+j_{k+1}=k}} \left(\frac{3C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}/2} \right)^k \\
\leq \left(\frac{4}{C} \right)^k \frac{6C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}}.$$
(128)

Step 4: bounding $\|U^{\text{oracle}}U^{\text{oracle}\top} - \widetilde{U}\widetilde{U}^{\top}\|_{2,\infty}$ and $\|U^{\text{oracle}\top} - U^{\star}U^{\star\top}\|$. By virtue of (127), (128) and Lemma 1, we reach

$$\left\| \boldsymbol{U}^{\mathsf{oracle}} \boldsymbol{U}^{\mathsf{oracle}\top} - \widetilde{\boldsymbol{U}} \widetilde{\boldsymbol{U}}^\top \right\|_{2,\infty} \quad \leq \sum_{1 \leq k \leq \log n} 8C_3 \sqrt{\frac{\mu r}{n_1}} \left(\frac{C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}} \right)^k$$

$$+\sum_{k\geq \lfloor \log n \rfloor +1} \left(\frac{4}{C}\right)^{k} \frac{6C_{5}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n}{\sigma_{r}^{\star 2}}$$
$$\lesssim \sqrt{\frac{\mu r}{n_{1}}} \frac{\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n}{\sigma_{r}^{\star 2}}.$$
(129)

In addition, the $\sin \Theta$ theorem (Chen et al., 2021b, Chapter 2) shows that

where the first identity makes use of Chen et al. (2021b, Lemma 2.5), the penultimate inequality results from Lemma 5, and the last relation comes from Assumption 2 and (83b). Moreover, applying (88d) and the previous inequality yields that

$$\begin{split} \|\widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{U}}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\|_{2,\infty} &\leq \left\| (\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}})\widetilde{\boldsymbol{U}}^{\top} \right\|_{2,\infty} + \left\| \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{U}}^{\top} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top} \right\|_{2,\infty} \\ &= \left\| (\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}})\widetilde{\boldsymbol{U}}^{\top} \right\|_{2,\infty} + \left\| \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}}_{\perp}\widetilde{\boldsymbol{U}}_{\perp}^{\top} \right\|_{2,\infty} \\ &\leq \left\| \widetilde{\boldsymbol{U}} - \boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}} \right\|_{2,\infty} + \left\| \boldsymbol{U}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}}_{\perp} \right\| \\ &\lesssim \frac{\sqrt{\mu r}\omega_{\max}\log n}{\sigma_{r}^{\star}} + \sqrt{\frac{\mu r}{n_{1}}}\frac{\sqrt{n_{1}}\omega_{\max}\log n}{\sigma_{r}^{\star}} \\ &\asymp \sqrt{\frac{\mu r}{n_{1}}}\frac{\sqrt{n_{1}}\omega_{\max}\log n}{\sigma_{r}^{\star}}. \end{split}$$

This taken collectively with (129) gives

where the last relation results from the assumption (83a).

Finally, the Davis-Kahan Theorem, (88a) and (88c) together show that

$$\begin{split} \left\| \boldsymbol{U}^{\mathsf{oracle}} \boldsymbol{U}^{\mathsf{oracle}\top} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star\top} \right\| &\leq \left\| \boldsymbol{U}^{\mathsf{oracle}} \boldsymbol{U}^{\mathsf{oracle}\top} - \widetilde{\boldsymbol{U}} \widetilde{\boldsymbol{U}}^{\top} \right\| + \left\| \widetilde{\boldsymbol{U}} \widetilde{\boldsymbol{U}}^{\top} - \boldsymbol{U}^{\star} \boldsymbol{U}^{\star\top} \right\| \\ &\lesssim \frac{\left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star 2}} + \frac{\sqrt{n_1} \omega_{\mathsf{max}} \log n}{\sigma_r^{\star}} \\ &\lesssim \frac{\sqrt{n_1 n_2} \omega_{\mathsf{max}}^2 \log^2 n}{\sigma_r^{\star 2}} + \frac{\sqrt{n_1} \omega_{\mathsf{max}} \log n}{\sigma_r^{\star}}. \end{split}$$

Here, we have used the triangle inequality in the first inequality, the second inequality comes from (130), the Davis-Kahan Theorem, (88b) and (88c), whereas the last inequality holds since

$$\frac{n_1 \omega_{\max}^2 \log^2 n}{\sigma_r^{\star 2}} \lesssim \frac{\sqrt{n_1} \omega_{\max} \log n}{\sigma_r^{\star}}$$

under our signal-to-noise condition (83a). This concludes the proof.

B.4 Proof of Lemma 2

To streamline the presentation, we divide the proof into several steps. We shall start by considering the case with bounded noise (i.e., the case with $|E_{i,j}| \leq B$ deterministically) and develop upper bounds on both $\| \left[\mathcal{P}_{\mathsf{off}-\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \|_{2,\infty}$ and $\| \boldsymbol{E}^{\top} \left[\mathcal{P}_{\mathsf{off}-\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \|_{2,\infty}$ via induction. We will then move on to the general case and establish the final result by means of a truncation trick.

B.4.1 The case with bounded noise

Let us now focus on the case where

$$|E_{i,j}| \le B \le C_{\mathsf{b}}\omega_{\mathsf{max}} \frac{\min\left\{ (n_1 n_2)^{1/4}, \sqrt{n_2} \right\}}{\log n}, \quad \forall (i,j) \in [n_1] \times [n_2]$$
(131)

holds deterministically. We would like to prove, by induction, the following slightly stronger claims: suppose that E satisfies Conditions 1 and 2 in Assumption 1 and (131), then for any $0 \le k \le \log n$, with probability exceeding $1 - O((n+3)^{2k}n^{-C_2 \log n})$ one has

$$\left\| \left[\mathcal{P}_{\mathsf{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \leq C_3 \sqrt{\frac{\mu r}{n_2}} \left(C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n \right)^{\ell} \sqrt{n_2} \omega_{\mathsf{max}} \log n$$
(132)

and

$$\left\| \boldsymbol{E}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \leq C_{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{\ell} \left(\sqrt{n_{2}} B \omega_{\mathsf{max}} \log n + \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \right) \log^{2} n$$
(133)

for all $0 \le \ell \le k$. Here, $C_3, C_4 > 0$ are some large numerical constants to be specified shortly.

Step 1: base case. Let us first look at the base case with k = 0. It follows from Lemma 6 and the assumption (131) that: for any fixed matrices W_1 with n_2 rows and any W_2 with n_1 rows, one has

$$\max_{i \in [n_1]} \sum_{j \in [n_2]} E_{i,j}^2 \lesssim B^2 \log^2 n + \omega_{\mathsf{row}}^2 \lesssim n_2 \omega_{\mathsf{max}}^2$$
(134a)

$$\max_{\in [n_1]} \left\| \boldsymbol{E}_{i,:} \boldsymbol{W}_1 \right\|_2 \lesssim B \left\| \boldsymbol{W}_1 \right\|_{2,\infty} \log^2 n + \omega_{\max} \left\| \boldsymbol{W}_1 \right\|_F \log n \lesssim \sqrt{n_2} \omega_{\max} \left\| \boldsymbol{W}_1 \right\|_{2,\infty} \log n$$
(134b)

$$\max_{j \in [n_2]} \sum_{i \in [n_1]} E_{i,j}^2 \lesssim B^2 \log^2 n + \omega_{\mathsf{col}}^2$$
(134c)

$$\max_{j \in [n_2]} \left\| \left(\boldsymbol{E}_{:,j} \right)^\top \boldsymbol{W}_2 \right\|_2 \lesssim \left(B \log^2 n + \omega_{\mathsf{col}} \log n \right) \left\| \boldsymbol{W}_2 \right\|_{2,\infty}$$
(134d)

with probability exceeding $1 - O(n^{-C_4 \log n})$ for some numerical constant $C_4 > 0$. Inequality (134b) combined with Definition 1 tells us that with probability at least $1 - O(n^{-C_4 \log n})$,

$$\|\boldsymbol{E}\boldsymbol{V}^{\star}\|_{2,\infty} \lesssim \sqrt{n_2}\omega_{\max} \|\boldsymbol{V}^{\star}\|_{2,\infty} \log n \le \sqrt{\mu r}\omega_{\max} \log n.$$
(135)

In addition, for any $j \in [n_2]$, we can decompose $E_{:,j}^{\top} E V^{\top}$ into two terms:

$$\boldsymbol{E}_{:,j}^{\top} \boldsymbol{E} \boldsymbol{V}^{\star} = \boldsymbol{E}_{:,j}^{\top} \boldsymbol{E}^{(:,-j)} \boldsymbol{V}^{\star} + \boldsymbol{E}_{:,j}^{\top} \boldsymbol{E}^{(:,j)} \boldsymbol{V}^{\star}.$$
(136)

Here, $\boldsymbol{E}^{(:,-j)}$ and $\boldsymbol{E}^{(:,j)}$ are defined as

$$\boldsymbol{E}^{(:,-j)} = \mathcal{P}_{:,-j}\left(\boldsymbol{E}\right) \in \mathbb{R}^{n_1 \times n_2} \quad ext{and} \quad \boldsymbol{E}^{(:,j)} = \mathcal{P}_{:,j}\left(\boldsymbol{E}\right) \in \mathbb{R}^{n_1 \times n_2},$$

where $\mathcal{P}_{:,-j}(\cdot)$ (resp. $\mathcal{P}_{:,j}(\cdot)$) is a projection operator that zeros out the *j*-th column (resp. all entries except those in the *j*-th column) of a matrix, i.e., for any matrix \boldsymbol{A} ,

$$\left[\mathcal{P}_{:,-j}(\boldsymbol{A})\right]_{i,k} = \begin{cases} A_{i,k}, & \text{if } k \neq j, \\ 0, & \text{otherwise,} \end{cases} \quad \forall (i,k) \in [n_1] \times [n_2], \text{ and } \mathcal{P}_{:,j}(\boldsymbol{A}) = \boldsymbol{A} - \mathcal{P}_{:,-j}(\boldsymbol{A}). \tag{137}$$

In view of (134b) and (134d), with probability exceeding $1 - O(n^{-C_2 \log n})$,

$$\left\|\boldsymbol{E}_{:,j}^{\top}\boldsymbol{E}^{(:,-j)}\boldsymbol{V}^{\star}\right\|_{2} \lesssim \left(B\log^{2}n + \omega_{\mathsf{col}}\log n\right)\left\|\boldsymbol{E}^{(:,-j)}\boldsymbol{V}^{\star}\right\|_{2,\infty}$$

$$\lesssim \left(B\log n + \omega_{\rm col}\right)\sqrt{\mu r}\omega_{\rm max}\log^2 n,$$

where the last inequality can be derived in a way similar to (135). Recognizing that $(\mathbf{E}_{:,j}^{\top} \mathbf{E}^{(:,j)})^{\top}$ is a vector with only one nonzero entry $\|\mathbf{E}_{:,j}\|_2^2$, we know from (134c) and Definition 1 that, with probability at least $1 - O(n^{-C_4 \log n})$,

$$\left\|\boldsymbol{E}_{:,j}^{\top}\boldsymbol{E}^{(:,j)}\boldsymbol{V}^{\star}\right\|_{2} \leq \left\|\boldsymbol{E}_{:,j}\right\|_{2}^{2}\left\|\boldsymbol{V}^{\star}\right\|_{2,\infty} \lesssim \left(B^{2}\log^{2}n + \omega_{\mathsf{col}}^{2}\right)\sqrt{\frac{\mu r}{n_{2}}}.$$

Taking the previous two inequalities and (136) together and applying the union bound imply that, with probability at least $1 - O(n^{-C_2 \log n})$,

$$\begin{split} \left\| \boldsymbol{E}^{\top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} &\lesssim \left(\sqrt{n_2} B \omega_{\max} \log^3 n + \sqrt{n_2} \omega_{\text{col}} \omega_{\max} \log^2 n + B^2 \log^2 n + \omega_{\text{col}}^2 \right) \sqrt{\frac{\mu r}{n_2}} \\ &\lesssim \sqrt{\frac{\mu r}{n_2}} \left(\sqrt{n_2} B \omega_{\max} \log n + \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\max}^2 \right) \log^2 n, \end{split}$$

where we have also made use of the assumption (131).

Therefore, we have established both (132) and (133) for the base case with k = 0.

Step 2: inductive step. We now move on to the inductive step. Suppose that for any E satisfying Conditions 1 and 2 in Assumption 1 and (131), the induction hypotheses (132) and (133) hold for all $1 \le \ell \le K$ with probability exceeding $1 - O((n+2)^{2K} \cdot n^{-C_2 \log n})$. We intend to justify that these induction hypotheses continue to be valid for K + 1.

Step 2.1: bounding $\|[\mathcal{P}_{off-diag}(\boldsymbol{E}\boldsymbol{E}^{\top})]^{K+1}\boldsymbol{E}\boldsymbol{V}^{\star}\|_{2,\infty}$. We first look at the quantity of interest in (132). For any $i \in [n_1]$, define

$$\boldsymbol{E}^{(-i,:)} = \mathcal{P}_{-i,:}(\boldsymbol{E}) \in \mathbb{R}^{n_1 \times n_2} \text{ and } \boldsymbol{E}^{(i,:)} = \mathcal{P}_{i,:}(\boldsymbol{E}) \in \mathbb{R}^{n_1 \times n_2}.$$

Here, $\mathcal{P}_{-i,:}(A)$ (resp. $\mathcal{P}_{i,:}(A)$) zeros out the *i*-th row (resp. all entries except the ones in the *i*-th row) of A, namely,

$$\left[\mathcal{P}_{-i,:}(\boldsymbol{A})\right]_{j,k} = \begin{cases} A_{j,k}, & \text{if } j \neq i, \\ 0, & \text{otherwise,} \end{cases} \quad \forall (j,k) \in [n_1] \times [n_2], \quad \text{and} \quad \mathcal{P}_{i,:}(\boldsymbol{A}) = \boldsymbol{A} - \mathcal{P}_{-i,:}(\boldsymbol{A}). \tag{138}$$

When it comes to k = K + 1, recognizing the identity

$$\left[\mathcal{P}_{\mathsf{off} ext{-diag}}\left(\boldsymbol{E}\boldsymbol{E}^{ op}
ight)
ight]_{i,:}=\boldsymbol{E}_{i,:}\boldsymbol{E}^{(-i,:) op},$$

we can derive

$$\left[\left[\mathcal{P}_{\mathsf{off}\mathsf{-diag}}\left(\boldsymbol{\boldsymbol{E}}\boldsymbol{\boldsymbol{E}}^{\top}\right)\right]^{K+1}\boldsymbol{\boldsymbol{E}}\boldsymbol{\boldsymbol{V}}^{\star}\right]_{i,:} = \boldsymbol{\boldsymbol{E}}_{i,:}\boldsymbol{\boldsymbol{E}}^{(-i,:)\top}\left[\mathcal{P}_{\mathsf{off}\mathsf{-diag}}\left(\boldsymbol{\boldsymbol{E}}\boldsymbol{\boldsymbol{E}}^{\top}\right)\right]^{K}\boldsymbol{\boldsymbol{E}}\boldsymbol{\boldsymbol{V}}^{\star}.$$
(139)

We claim for the moment that

$$\begin{aligned} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2} \\ \leq \underbrace{\left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{E}^{(-i,:)} \boldsymbol{V}^{\star} \right\|_{2}}_{=:\tau_{1}} \\ + \underbrace{\sum_{\ell=0}^{K-1} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \right\|_{2}^{2} \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right\|^{\ell} \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty}}_{=:\tau_{2}}, \quad (140) \end{aligned}$$

which we shall prove towards the end of the proof for the bounded noise case. We define the following event

$$\begin{split} \mathcal{E}_{1} &= \bigg\{ \forall 0 \leq \ell \leq K - 1, \quad \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ &\leq C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{\ell} \sqrt{n_{2}} \omega_{\max} \log n, \\ \forall i \in [n_{1}], \quad \left\| \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{E}^{(-i,:)} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ &\leq C_{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{K} \left(\sqrt{n_{2}} B \omega_{\max} \log n + \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \right) \log^{2} n \bigg\}. \end{split}$$

Recognizing that $E^{(-i,:)}$ satisfies Conditions 1 and 2 in Assumption 1 and (131) as well, we learn from our induction hypotheses and the union bound that

$$\mathbb{P}(\mathcal{E}_1) \ge 1 - (n_1 + 1) \cdot C_1 (n + 3)^{2K} n^{-C_2 \log n}$$

Moreover, Lemma 7 asserts that with probability exceeding $1 - O(n^{-C_2 \log n})$,

$$\left\|\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top}\right)\right\| \leq C_5 \left(\sqrt{n_1 n_2} + n_1\right) \omega_{\mathsf{max}}^2 \log^2 n.$$
(141)

Given that $E_{i,:}E^{(-i,:)\top}$ is the *i*-th row of $\mathcal{P}_{\text{off-diag}}(EE^{\top})$ and $\mathcal{P}_{\text{off-diag}}(E^{(-i,:)}E^{(-i,:)\top})$ is a submatrix of $\mathcal{P}_{\text{off-diag}}(EE^{\top})$, the inequality (141) implies that

$$\max\left\{\left\|\boldsymbol{E}_{i,:}\boldsymbol{E}^{(-i,:)\top}\right\|_{2}, \left\|\mathcal{P}_{\mathsf{off}\mathsf{-diag}}\left(\boldsymbol{E}^{(-i,:)}\boldsymbol{E}^{(-i,:)\top}\right)\right\|\right\} \le C_{5}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\mathsf{max}}^{2}\log^{2}n.$$
(142)

Armed with these results, we proceed to bound τ_1 and τ_2 in (140) separately in the sequel.

• Bounding τ_1 . Note that $\boldsymbol{E}^{(-i,:)\top}[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\boldsymbol{E}^{(-i,:)}\boldsymbol{E}^{(-i,:)\top})]^K \boldsymbol{E}^{(-i,:)} \boldsymbol{V}^{\star}$ is statistically independent of $\boldsymbol{E}_{i,:}$. In view of (134b), with probability exceeding $1 - O(n^{-C_2 \log n})$, one has

$$\tau_{1} \leq C_{5}B \left\| \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{E}^{(-i,:)} \boldsymbol{V}^{\star} \right\|_{2,\infty} \log^{2} n + C_{5}\omega_{\mathsf{max}} \left\| \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{E}^{(-i,:)} \boldsymbol{V}^{\star} \right\|_{\mathrm{F}} \log n$$
(143)

for some suitable universal constants $C_2, C_5 > 0$. We have also learned from Lemma 5 that

$$\left\|\boldsymbol{E}^{(-i,:)}\boldsymbol{V}^{\star}\right\| \leq \left\|\boldsymbol{E}\boldsymbol{V}^{\star}\right\| \leq C_{5}\left(B\sqrt{\frac{\mu r}{n_{2}}}\log^{2}n + \sqrt{n_{1}}\omega_{\max}\log n\right)$$
(144)

and

$$\left\|\boldsymbol{E}^{(-i,:)}\right\| \le \left\|\boldsymbol{E}\right\| \le C_5 \left(\sqrt{n_2}\omega_{\max} + \sqrt{n_1}\omega_{\max}\right)$$
(145)

with probability exceeding $1 - O(n^{-C_2 \log n})$, provided that C_5 is large enough. Let \mathcal{E}_2 denote the event $\mathcal{E}_2 = \{(141), (143), (144) \text{ and } (145) \text{ hold}\}$. Then $\mathbb{P}(\mathcal{E}_2) \geq 1 - O(n^{-C_2 \log n})$ and, consequently,

$$\mathcal{P}\left(\mathcal{E}_{1} \cap \mathcal{E}_{2}\right) \ge 1 - C_{1}(n_{1} + 2)(n + 3)^{2K} n^{-C_{2}\log n}.$$
(146)

On the event \mathcal{E}_1 , one has

$$\begin{split} \left\| \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{E}^{(-i,:)} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ & \leq C_{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{K} \left(\sqrt{n_{2}} B \omega_{\max} \log n + \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \right) \log^{2} n. \end{split}$$

In view of (142), (143), (144), (145), the previous inequality and the assumption (131), on the event $\mathcal{E}_1 \cap \mathcal{E}_2$ we have

$$\begin{split} \tau_{1} &\leq C_{5}B \cdot C_{4}\sqrt{\frac{\mu r}{n_{2}}} \left(C_{3}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n\right)^{K}\left(\sqrt{n_{2}}B\omega_{\max}\log n+\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\right)\log^{4}n \\ &+\sqrt{r} \cdot C_{5}\omega_{\max} \cdot \left\|\boldsymbol{E}^{(-i,:)}\right\| \left\|\mathcal{P}_{\text{off-diag}}\left(\boldsymbol{E}^{(-i,:)}\boldsymbol{E}^{(-i,:)\top}\right)\right\|^{K} \left\|\boldsymbol{E}^{(-i,:)}\boldsymbol{V}^{\star}\right\|\log n \\ &\leq C_{4}\sqrt{\frac{\mu r}{n_{2}}} \left(C_{3}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n\right)^{K} \left(C_{5}C_{b}^{2}\sqrt{n_{2}}\sqrt{n_{1}n_{2}}\omega_{\max}^{3}+C_{5}C_{b}\sqrt{n_{2}}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{3}\right)\log^{3}n \\ &+\sqrt{r} \cdot C_{5}\omega_{\max} \cdot C_{5}\left(\sqrt{n_{1}}+\sqrt{n_{2}}\right)\omega_{\max} \cdot \left(C_{5}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n\right)^{K} \\ &\cdot C_{5}\left(B\sqrt{\frac{\mu r}{n_{2}}}\log^{2}n+\sqrt{n_{1}}\omega_{\max}\log n\right)\log n \\ &\leq \frac{C_{3}}{4}\sqrt{\frac{\mu r}{n_{2}}}\left(C_{3}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n\right)^{K+1}\sqrt{n_{2}}\omega_{\max}\log n \\ &+C_{5}^{2}\sqrt{r} \cdot \left(\sqrt{n_{1}}+\sqrt{n_{2}}\right)\omega_{\max}^{2} \cdot \left(C_{5}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n\right)^{K} \cdot C_{5}(C_{b}+1)\sqrt{\mu n_{1}}\omega_{\max}\log^{2}n \\ &\leq \frac{C_{3}}{2}\sqrt{\frac{\mu r}{n_{2}}}\left(C_{3}\left(\sqrt{n_{1}n_{2}}+n_{1}\right)\omega_{\max}^{2}\log^{2}n\right)^{K+1}\sqrt{n_{2}}\omega_{\max}\log n, \end{split}$$

provided that $C_3^2 \ge 4C_4(C_5C_b^2 + C_5C_b + (C_b + 1)C_5^2)$. Here, the second and the third inequalities are due to the assumption (131).

• Bounding τ_2 . By virtue of (142) and the induction hypotheses, on the same event $\mathcal{E}_1 \cap \mathcal{E}_2$ we have

$$\begin{aligned} \tau_{2} &\leq \sum_{\ell=0}^{K-1} \left[C_{5} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right]^{\ell+2} \cdot C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{K-1-\ell} \sqrt{n_{2}} \omega_{\max} \log n \\ &\leq \sum_{\ell=0}^{K-1} \frac{1}{2^{\ell+2}} C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{K+1} \sqrt{n_{2}} \omega_{\max} \log n \\ &\leq \frac{C_{3}}{2} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\max}^{2} \log^{2} n \right)^{K+1} \sqrt{n_{2}} \omega_{\max} \log n, \end{aligned}$$

with the proviso that $C_3 \ge 2C_5$.

Putting the previous bounds on τ_1 and τ_2 together with (139) and (140), we arrive at the following result that holds on the event $\mathcal{E}_1 \cap \mathcal{E}_2$:

$$\left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K+1} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \le C_3 \sqrt{\frac{\mu r}{n_2}} \left(C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n \right)^{K+1} \sqrt{n_2} \omega_{\mathsf{max}} \log n,$$
(147)

provided that $C_3^2 \ge 4C_4(C_5C_b^2 + C_5C_b + (C_b + 1)C_5^2)$ and $C_3 \ge 2C_5$.

Step 2.2: bounding $\|E^{\top}[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(EE^{\top})]^{K+1}EV^{\star}\|_{2,\infty}$. We then move on to the quantity of interest in (133). For any $j \in [n_2]$, it can be easily verified that

$$\left(\boldsymbol{E}^{\top}\left[\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top}\right)\right]^{K+1}\boldsymbol{E}\boldsymbol{V}^{\star}\right)_{j,:} = \boldsymbol{E}_{:,j}^{\top}\left[\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top}\right)\right]^{K+1}\boldsymbol{E}\boldsymbol{V}^{\star}.$$
(148)

Recalling that $E^{(:,-j)} = \mathcal{P}_{:,-j}(E)$ and $E^{(:,j)} = \mathcal{P}_{:,j}(E)$, we have

$$\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top}\right) = \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\big(\boldsymbol{E}^{(:,j)}\boldsymbol{E}^{(:,j)\top}\big) + \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\big(\boldsymbol{E}^{(:,-j)}\boldsymbol{E}^{(:,-j)\top}\big)$$

For any matrices $A, B \in \mathbb{R}^{n_1 \times n_1}$, it is straightforward to show that

$$(A + B)^{K+1} = B^{K+1} + \sum_{\ell=0}^{K} B^{\ell} A (A + B)^{K-\ell},$$

and consequently one has

$$\begin{split} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right]^{K+1} &= \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top} \right) \right]^{K+1} \\ &+ \sum_{\ell=0}^{K} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top} \right) \right]^{\ell} \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,j)} \boldsymbol{E}^{(:,j)\top} \right) \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right]^{K-\ell}. \end{split}$$

As a result, we can express $\boldsymbol{E}_{:,j}^{\top} [\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\boldsymbol{E}\boldsymbol{E}^{\top})]^{K+1} \boldsymbol{E} \boldsymbol{V}^{\star}$ in terms of a sum of vectors as follows:

$$E_{:,j}^{\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\mathsf{T}} \right) \right]^{K+1} \boldsymbol{E} \boldsymbol{V}^{\star}$$

$$= E_{:,j}^{\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\mathsf{T}} \right) \right]^{K+1} \boldsymbol{E} \boldsymbol{V}^{\star}$$

$$+ \sum_{\ell=0}^{K} E_{:,j}^{\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\mathsf{T}} \right) \right]^{\ell} \mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,j)} \boldsymbol{E}^{(:,j)\mathsf{T}} \right) \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\mathsf{T}} \right) \right]^{K-\ell} \boldsymbol{E} \boldsymbol{V}^{\star}$$

$$= \underbrace{E_{:,j}^{\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\mathsf{T}} \right) \right]^{K+1} \boldsymbol{E}^{(:,-j)} \boldsymbol{V}^{\star}}_{=:b_{1}}$$

$$+ \underbrace{E_{:,j}^{\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\mathsf{T}} \right) \right]^{K+1} \boldsymbol{E}^{(:,j)} \boldsymbol{V}^{\star}}_{=:b_{2}}$$

$$+ \underbrace{\sum_{\ell=0}^{K} E_{:,j}^{\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\mathsf{T}} \right) \right]^{\ell} \mathcal{P}_{\text{diag}} \left(\boldsymbol{E}^{(:,j)\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\mathsf{T}} \right) \right]^{K-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right]$$

$$=:b_{3}$$

$$- \underbrace{\sum_{\ell=0}^{K} E_{:,j}^{\mathsf{T}} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\mathsf{T}} \right) \right]^{\ell} \mathcal{P}_{\text{diag}} \left(\boldsymbol{E}^{(:,j)\mathsf{T}} \right) \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\mathsf{T}} \right) \right]^{K-\ell} \boldsymbol{E} \boldsymbol{V}^{\star}, \quad (149)$$

thus motivating us to bound each of these terms $\|\boldsymbol{b}_1\|_2$, $\|\boldsymbol{b}_2\|_2$, $\|\boldsymbol{b}_3\|_2$ and $\|\boldsymbol{b}_4\|_2$ separately. Let \mathcal{E}_3 denote the following event:

$$\mathcal{E}_{3} = \left\{ \forall 0 \leq \ell \leq K, \left\| \boldsymbol{E}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{E}\boldsymbol{V}^{\star} \right\|_{2,\infty} \\ \leq C_{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{\ell} \left(\sqrt{n_{2}} B \omega_{\mathsf{max}} \log n + \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \right) \log^{2} n, \\ \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{E}\boldsymbol{V}^{\star} \right\|_{2,\infty} \leq C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{\ell} \sqrt{n_{2}} \omega_{\mathsf{max}} \log n, \\ \forall j \in [n_{2}], \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top} \right) \right]^{K+1} \boldsymbol{E}^{(:,-j)} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ \leq C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{K+1} \sqrt{n_{2}} \omega_{\mathsf{max}} \log n \right\}.$$
(150)

The induction hypotheses and (147) taken together with the union bound indicate that

$$\mathbb{P}(\mathcal{E}_3) \ge 1 - C_1(n_2 + 1)(n_1 + 2)(n + 3)^{2K} n^{-C_2 \log n}.$$

By virtue of (134c), (134d) and the independence between $[\mathcal{P}_{\mathsf{off}-\mathsf{diag}}(\boldsymbol{E}^{(:,-j)}\boldsymbol{E}^{(:,-j)\top})]^{K+1}\boldsymbol{E}^{(:,-j)}\boldsymbol{V}^{\star}$ and $\boldsymbol{E}_{:,j}$, one has, with probability exceeding $1 - O(n^{-C_2 \log n})$,

$$\max_{j \in [n_2]} \|\boldsymbol{E}_{:,j}\|_2^2 \le C_5 \left(B^2 \log^2 n + \omega_{\mathsf{col}}^2 \right)$$
(151)

and

$$\|\boldsymbol{b}_1\|_2 \le C_5 \left(B\log^2 n + \omega_{\mathsf{col}}\log n\right) \left\| \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top} \right) \right]^{K+1} \boldsymbol{E}^{(:,-j)} \boldsymbol{V}^* \right\|_{2,\infty}.$$
 (152)

Applying Lemma 7 and the union bound yields that with probability exceeding $1 - O(n^{-C_2 \log n})$,

$$\left\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\boldsymbol{E}^{(:,-j)}\boldsymbol{E}^{(:,-j)\top}\right)\right\| \le C_5 \left(\sqrt{n_1 n_2} + n_1\right) \omega_{\mathsf{max}}^2 \log^2 n \tag{153}$$

for all $j \in [n_2]$. Let $\mathcal{E}_4 = \{(151), (152) \text{ and } (153) \text{ hold} \}$ and $\mathcal{E}_5 = \mathcal{E}_3 \cap \mathcal{E}_4$. Thus, $\mathbb{P}(\mathcal{E}_4) \ge 1 - O(n^{-C_2 \log n})$, and as a result,

$$\mathbb{P}(\mathcal{E}_5) \ge 1 - C_1(n+2)^2(n+3)^{2K} n^{-C_2 \log n}.$$

Armed with these events, we shall bound b_1, \ldots, b_5 separately in what follows.

• Bounding $\|\boldsymbol{b}_1\|_2$. In view of (152), (150) and Assumption 2, we know that on the event \mathcal{E}_5 ,

$$\begin{aligned} \|\boldsymbol{b}_{1}\|_{2} &\leq C_{5} \left(B \log^{2} n + \omega_{\mathsf{col}} \log n\right) \cdot C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1} n_{2}} + n_{1}\right) \omega_{\mathsf{max}}^{2} \log^{2} n\right)^{K+1} \sqrt{n_{2}} \omega_{\mathsf{max}} \log n \\ &\leq C_{3} C_{5} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1} n_{2}} + n_{1}\right) \omega_{\mathsf{max}}^{2} \log^{2} n\right)^{K+1} \left(\sqrt{n_{2}} B \omega_{\mathsf{max}} \log n + \sqrt{n_{1} n_{2}} \omega_{\mathsf{max}}^{2}\right) \log^{2} n \\ &\leq \frac{C_{4}}{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1} n_{2}} + n_{1}\right) \omega_{\mathsf{max}}^{2} \log^{2} n\right)^{K+1} \left(\sqrt{n_{2}} B \omega_{\mathsf{max}} \log n + \sqrt{n_{1} n_{2}} \omega_{\mathsf{max}}^{2}\right) \log^{2} n, \end{aligned}$$
(154)

as long as $C_4 \ge 4C_3C_5$.

• Bounding $\|\boldsymbol{b}_2\|_2$. Turning to \boldsymbol{b}_2 , we recognize that $\boldsymbol{E}_{:,j}^{\top}[\mathcal{P}_{\mathsf{off}\mathsf{-diag}}(\boldsymbol{E}^{(:,-j)}\boldsymbol{E}^{(:,-j)\top})]^{K+1}\boldsymbol{E}^{(:,j)}$ is a vector with only one nonzero entry $\boldsymbol{E}_{:,j}^{\top}[\mathcal{P}_{\mathsf{off}\mathsf{-diag}}(\boldsymbol{E}^{(:,-j)}\boldsymbol{E}^{(:,-j)\top})]^{K+1}\boldsymbol{E}_{:,j}$. By virtue of (151), (153) and the assumption (131), one sees that on the event \mathcal{E}_5 ,

$$\begin{aligned} \|\boldsymbol{b}_{2}\|_{2} &\leq \left|\boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top}\right)\right]^{K+1} \boldsymbol{E}_{:,j}\right| \|\boldsymbol{V}^{\star}\|_{2,\infty} \\ &\leq \|\boldsymbol{E}_{:,j}\|_{2}^{2} \left\|\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top}\right)\right\|^{K+1} \sqrt{\frac{\mu r}{n_{2}}} \\ &\leq C_{5} \left(B^{2} \log^{2} n + \omega_{\text{col}}^{2}\right) \cdot \left(C_{5} \left(\sqrt{n_{1} n_{2}} + n_{1}\right) \omega_{\text{max}}^{2} \log^{2} n\right)^{K+1} \sqrt{\frac{\mu r}{n_{2}}} \\ &\leq C_{5} (C_{b} + 1) \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1} n_{2}} + n_{1}\right) \omega_{\text{max}}^{2} \log^{2} n\right)^{K+1} \left(\sqrt{n_{2}} B \omega_{\text{max}} \log n + n_{1} \omega_{\text{max}}^{2}\right) \log^{2} n \\ &\leq \frac{C_{4}}{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1} n_{2}} + n_{1}\right) \omega_{\text{max}}^{2} \log^{2} n\right)^{K+1} \left(\sqrt{n_{2}} B \omega_{\text{max}} \log n + \left(\sqrt{n_{1} n_{2}} + n_{1}\right) \omega_{\text{max}}^{2}\right) \log^{2} n, \end{aligned}$$
(155)

provided that $C_4 \ge 4C_5(C_b + 1)$.

• Bounding $\|\mathbf{b}_3\|_2$. With regards to \mathbf{b}_3 , repeating a similar argument as for (155) shows that on the same event, it holds that

$$\begin{split} \|\boldsymbol{b}_{3}\|_{2} &\leq \sum_{\ell=0}^{K} \left|\boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top}\right)\right]^{\ell} \boldsymbol{E}_{:,j} \right| \left\|\boldsymbol{E}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top}\right)\right]^{K-\ell} \boldsymbol{E}\boldsymbol{V}^{\star} \right\|_{2,\infty} \\ &\leq \sum_{\ell=0}^{K} \left\|\boldsymbol{E}_{:,j}\right\|_{2}^{2} \left\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top}\right)\right\|^{\ell} \cdot C_{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\max}^{2} \log^{2} n\right)^{K-\ell} \\ &\cdot \left(\sqrt{n_{2}} B \omega_{\max} \log n + \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\max}^{2}\right) \log^{2} n \\ &\leq \sum_{\ell=0}^{K} C_{5} \left(B^{2} \log^{2} n + \omega_{\mathsf{col}}^{2}\right) \cdot \left(C_{5} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\max}^{2} \log^{2} n\right)^{\ell} \cdot C_{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\max}^{2} \log^{2} n\right)^{K-\ell} \end{split}$$

$$\cdot \left(\sqrt{n_2}B\omega_{\max}\log n + (\sqrt{n_1n_2} + n_1)\omega_{\max}^2\right)\log^2 n$$

$$\leq \sum_{\ell=0}^{K} \frac{1}{2^{\ell}}C_4C_5\left(C_b^2\sqrt{n_1n_2}\omega_{\max}^2 + n_1\omega_{\max}^2\right)\sqrt{\frac{\mu r}{n_2}}\left(C_3\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log^2 n\right)^K$$

$$\cdot \left(\sqrt{n_2}B\omega_{\max}\log n + (\sqrt{n_1n_2} + n_1)\omega_{\max}^2\right)\log^2 n$$

$$\leq 2C_4C_5\left(C_b^2 + 1\right)C_3^{-1}\sqrt{\frac{\mu r}{n_2}}\left(C_3\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log^2 n\right)^{K+1}$$

$$\cdot \left(\sqrt{n_2}B\omega_{\max}\log n + (\sqrt{n_1n_2} + n_1)\omega_{\max}^2\right)\log^2 n$$

$$\leq \frac{C_4}{4}\sqrt{\frac{\mu r}{n_2}}\left(C_3\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log^2 n\right)^{K+1}\left(\sqrt{n_2}B\omega_{\max}\log n + (\sqrt{n_1n_2} + n_1)\omega_{\max}^2\right)\log^2 n, \quad (156)$$

with the proviso that $C_3 \ge 8C_5(C_b^2 + 1)$.

• Bounding $\|\boldsymbol{b}_4\|_2$. Regarding \boldsymbol{b}_4 , using the elementary bound $\|\boldsymbol{a}^\top \boldsymbol{B}\|_2 \leq \|\boldsymbol{a}\|_1 \|\boldsymbol{B}\|_{2,\infty}$ for any vector \boldsymbol{a} and matrix \boldsymbol{B} and applying (151), (153) and (150), we can demonstrate that on the event \mathcal{E}_5 ,

$$\begin{split} \|\boldsymbol{b}_{4}\|_{2} &\leq \sum_{\ell=0}^{K} \left\|\boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top}\right)\right]^{\ell} \mathcal{P}_{\text{diag}} \left(\boldsymbol{E}^{(:,j)} \boldsymbol{E}^{(:,j)\top}\right) \right\|_{1} \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top}\right)\right]^{K-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ &\leq \sum_{\ell=0}^{K} \left\|\boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top}\right)\right]^{\ell} \right\|_{2} \left\|\boldsymbol{E}_{:,j}\right\|_{2}^{2} \left\| \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top}\right)\right]^{K-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ &\leq \sum_{\ell=0}^{K} \sqrt{C_{5}} \left(B \log n + \omega_{\text{col}}\right) \cdot \left(C_{5} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\text{max}}^{2} \log^{2} n\right)^{\ell} \cdot C_{5} \left(B^{2} \log^{2} n + \omega_{\text{col}}^{2}\right) \\ &\quad \cdot C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\text{max}}^{2} \log^{2} n\right)^{K-\ell} \sqrt{n_{2}} \omega_{\text{max}} \log n \\ &\leq \sum_{\ell=0}^{K} \frac{1}{2^{\ell+1}} C_{3} \sqrt{C_{5}} \left(\sqrt{n_{2}} B \omega_{\text{max}} \log n + \sqrt{n_{1}n_{2}} \omega_{\text{max}}^{2}\right) \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\text{max}}^{2} \log^{2} n\right)^{K+1} \log n \\ &\leq \frac{C_{4}}{4} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\text{max}}^{2} \log^{2} n\right)^{K+1} \left(\sqrt{n_{2}} B \omega_{\text{max}} \log n + \left(\sqrt{n_{1}n_{2}} + n_{1}\right) \omega_{\text{max}}^{2}\right) \log^{2} n, \end{aligned}$$

$$\tag{157}$$

provided that $C_3 \ge 2C_5$ and $C_4 \ge 4C_3\sqrt{C_5}$.

Combine (149), (154), (155), (156) and (157) to reach that: on the \mathcal{E}_5 one has

$$\left\| \boldsymbol{E}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K+1} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty}$$

$$= \max_{j \in [n_2]} \left\| \boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K+1} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2}$$

$$\leq C_4 \sqrt{\frac{\mu r}{n_2}} \left(C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n \right)^{K+1} \left(\sqrt{n_2} B \omega_{\mathsf{max}} \log n + \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \right) \log^2 n, \quad (158)$$

with the proviso that $C_3 \ge 8C_5(C_b^2 + 1)$ and $C_4 \ge 4C_3C_5$. In summary, if the claim (140) is valid, then with probability exceeding $1 - C_1(n+2)(n+3)^{2k}n^{-C_2\log n}$, (147) and (158) hold simultaneously as long as $C_4 = 4C_3C_5$ and $C_3 \ge 32C_5^2(C_b^2 + 1)$. We have thus finished the proof of the induction hypotheses (132) and (133), as long as the claim (140) can be justified; see below.

Proof of the claim (140). We first make the observation that

$$\boldsymbol{E}^{(i,:)\top} \boldsymbol{E}^{(-i,:)} = \boldsymbol{E}^{(-i,:)\top} \boldsymbol{E}^{(i,:)} = \boldsymbol{0},$$
(159a)

$$\mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{E}^{(i,:)}\boldsymbol{E}^{(-i,:)\top}\right) = \mathcal{P}_{\mathsf{diag}}\left(\boldsymbol{E}^{(-i,:)}\boldsymbol{E}^{(i,:)\top}\right) = \mathbf{0},\tag{159b}$$

$$\boldsymbol{E}^{(i,:)\top} \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) = \boldsymbol{E}^{(-i,:)\top} \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{E}^{(i,:)} \boldsymbol{E}^{(i,:)\top} \right) = \boldsymbol{0}, \tag{159c}$$

$$\mathcal{P}_{\mathsf{diag}}(\boldsymbol{E}^{(-i,:)}\boldsymbol{E}^{(-i,:)\top})\mathcal{P}_{\mathsf{diag}}(\boldsymbol{E}^{(i,:)}\boldsymbol{E}^{(i,:)\top}) = \boldsymbol{0}.$$
(159d)

The identities (159a), (159b) and (159c) taken collectively give

$$\mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top}\right) = \mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{E}^{(-i,:)\top}\boldsymbol{E}^{(-i,:)\top}\right) + \mathcal{P}_{\mathsf{off-diag}}\left(\boldsymbol{E}^{(i,:)\top}\boldsymbol{E}^{(i,:)\top}\right) + \boldsymbol{E}^{(i,:)\top}\boldsymbol{E}^{(-i,:)\top} + \boldsymbol{E}^{(-i,:)\top}\boldsymbol{E}^{(i,:)\top}$$
(160) and

$$\begin{aligned} \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &= \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right] \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &+ \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \boldsymbol{E}^{(-i,:)\top} \boldsymbol{E}^{(i,:)\top} \left[\mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1} \boldsymbol{E} \boldsymbol{V}^{\star}. \end{aligned}$$
(161)

Combining (159a)-(159d) and (160) then yields

$$\begin{split} & \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right] \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right] \\ &= \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^2 + \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right] \boldsymbol{E}^{(-i,:)\top} \boldsymbol{E}^{(i,:)\top} \\ &+ \mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(i,:)} \boldsymbol{E}^{(i,:)\top} \right) \\ &+ \left[\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} - \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)\top} \right) \right] \boldsymbol{E}^{(i,:)} \boldsymbol{E}^{(-i,:)\top} \\ &= \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^2 + \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right] \boldsymbol{E}^{(-i,:)\top} \end{split}$$

As a consequence, we can deduce that

$$\begin{split} & \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &= \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{2} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-2} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &+ \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &+ \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right] \boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-2} \boldsymbol{E} \boldsymbol{V}^{\star}. \end{split}$$

Repeating the same argument yields

$$\begin{aligned} \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &= \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &+ \sum_{\ell=0}^{K-1} \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{\ell} \boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \\ &= \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{E}^{(-i,:)} \boldsymbol{V}^{\star} \\ &+ \sum_{\ell=0}^{K-1} \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{\ell} \boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1-\ell} \boldsymbol{E} \boldsymbol{V}^{\star}. \end{aligned}$$
(162)

Since $E_{i,:}E^{(-i,:)\top}[\mathcal{P}_{\mathsf{off}-\mathsf{diag}}(E^{(-i,:)}E^{(-i,:)\top})]^{\ell}E^{(-i,:)\top}$ is a vector with only one nonzero entry

$$\boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} [\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top})]^{\ell} \boldsymbol{E}^{(-i,:)} \boldsymbol{E}_{i,:}^{\top},$$

for any $0 \le \ell \le K - 1$, one can immediately derive

$$\begin{split} & \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{\ell} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2} \\ & \leq \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{\ell} \boldsymbol{E}^{(-i,:)} \boldsymbol{E}_{i,:}^{\top} \right\| \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ & \leq \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \right\|_{2}^{2} \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right\|^{\ell} \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1-\ell} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty}. \end{split}$$

Taking this together with (162) and the triangle inequality establishes the advertised result (140).

B.4.2 The general case

Having established the claim for the bounded noise case, we can readily turn attention to the more general case with the noise matrix E satisfying Assumption 2. To tackle this scenario, we introduce a properly truncated version $\tilde{E} = [\tilde{E}_{i,j}]_{(i,j)\in[n_1]\times[n_2]}$, which is a zero-mean matrix with entries given by

$$\widetilde{E}_{i,j} = E_{i,j} \mathbb{1}_{\{|E_{i,j}| \le B\}} - \mathbb{E} \left[E_{i,j} \mathbb{1}_{\{|E_{i,j}| \le B\}} \right].$$
(163)

It is clearly seen that

$$\operatorname{Var}\left[\widetilde{E}_{i,j}\right] \leq \mathbb{E}\left[\left(E_{i,j}\mathbbm{1}_{\{|E_{i,j}| \leq B\}}\right)^2\right] \leq \mathbb{E}\left[E_{i,j}^2\right] \leq \omega_{\max}^2$$

and

 $\left|\widetilde{E}_{i,j}\right| \le 2B.$

Then (132) and (133) tell us that with probability $1 - O(n^{-c_1 \log n})$,

$$\left\| \left[\mathcal{P}_{\mathsf{off}\mathsf{-diag}} \left(\widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top} \right) \right]^{k} \widetilde{\boldsymbol{E}} \boldsymbol{V}^{\star} \right\|_{2,\infty} \leq C_{3} \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{k} \sqrt{n_{2}} \omega_{\mathsf{max}} \log n$$
(164)

holds for all $0 \le k \le \log n$.

Let \overline{E} be another matrix whose entries are given by

$$\overline{E}_{i,j} = E_{i,j} \mathbb{1}_{\{|E_{i,j}| \le B\}}$$

In view of the Cauchy-Schwarz inequality, one can derive

$$\left\|\widetilde{\boldsymbol{E}} - \overline{\boldsymbol{E}}\right\| \le \left\|\widetilde{\boldsymbol{E}} - \overline{\boldsymbol{E}}\right\|_{\mathrm{F}} \le \sqrt{n_1 n_2} \max_{i,j} \left|\mathbb{E}\left[E_{i,j} \mathbb{1}_{\{|E_{i,j}| \le B\}}\right]\right| \le \sqrt{n_1 n_2} \left(\mathbb{E}\left[E_{i,j}^2\right] \mathbb{P}\left(|E_{i,j}| \le B\right)\right)^{1/2} \le \frac{\omega_{\max}}{n^5}.$$
(165)

By virtue of Lemmas 5 and 7, we can see that, with probability exceeding $1 - O(n^{-10})$,

$$\left\| \widetilde{E} \right\| \lesssim B\sqrt{\log n} + \omega_{\text{col}} + \omega_{\text{row}} \lesssim \sqrt{n}\omega_{\text{max}}$$
 (166)

and

$$\left\|\mathcal{P}_{\mathsf{off-diag}}\left(\widetilde{\boldsymbol{E}}\widetilde{\boldsymbol{E}}^{\top}\right)\right\| \lesssim B^2 \log^2 n + \omega_{\mathsf{col}}\left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}}\right) \log n.$$
(167)

Combining the above results reveals that, with probability exceeding $1 - O(n^{-10})$,

$$\|\overline{E}\| \leq \|\widetilde{E}\| + \|\widetilde{E} - \overline{E}\| \leq \sqrt{n}\omega_{\max},$$

$$\left\|\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\overline{\boldsymbol{E}}\,\overline{\boldsymbol{E}}^{\top}\right) - \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}\left(\widetilde{\boldsymbol{E}}\,\widetilde{\boldsymbol{E}}^{\top}\right)\right\| \leq 2\left\|\overline{\boldsymbol{E}}\,\overline{\boldsymbol{E}}^{\top} - \widetilde{\boldsymbol{E}}\,\widetilde{\boldsymbol{E}}^{\top}\right\| \leq 4\left\|\widetilde{\boldsymbol{E}} - \overline{\boldsymbol{E}}\right\| \left\|\widetilde{\boldsymbol{E}}\right\| + 2\left\|\widetilde{\boldsymbol{E}} - \overline{\boldsymbol{E}}\right\|^{2} \lesssim \frac{\omega_{\max}^{2}}{n^{4.5}},$$

and for all $0 \le k \le \log n$,

$$\begin{split} \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\overline{\boldsymbol{E}} \, \overline{\boldsymbol{E}}^{\top}) \right]^{k} \overline{\boldsymbol{E}} \boldsymbol{V}^{\star} - \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top}) \right]^{k} \widetilde{\boldsymbol{E}} \boldsymbol{V}^{\star} \right\|_{2,\infty} \\ & \leq \sum_{\ell=0}^{k-1} \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top})^{\ell} \left(\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\overline{\boldsymbol{E}} \, \overline{\boldsymbol{E}}^{\top}) - \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top}) \right) \right] \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\overline{\boldsymbol{E}} \, \overline{\boldsymbol{E}}^{\top}) \right]^{k-1-\ell} \overline{\boldsymbol{E}} \boldsymbol{V}^{\star} \right\| \\ & + \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}}(\widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top}) \right]^{k} \left(\overline{\boldsymbol{E}} - \widetilde{\boldsymbol{E}} \right) \boldsymbol{V}^{\star} \right\| \\ & \leq \sum_{\ell=0}^{k-1} \left[C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log n \right]^{\ell} \cdot C_{3} \frac{\omega_{\mathsf{max}}^{2}}{n^{4.5}} \cdot \left[C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log n \right]^{k-1-\ell} \cdot C_{3} \sqrt{n} \omega_{\mathsf{max}} \right] \end{split}$$

$$+ \left[C_3\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log n\right]^k \cdot \frac{\omega_{\max}}{n^5} \\ \leq (k+1)\left[C_3\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log n\right]^k \cdot \frac{\omega_{\max}}{n^4} \\ \ll \sqrt{\frac{\mu r}{n_2}}\left(C_3\left(\sqrt{n_1n_2} + n_1\right)\omega_{\max}^2\log^2 n\right)^k \sqrt{n_2}\omega_{\max}\log n.$$

Taking this collectively with (164) implies that, with probability exceeding $1 - O(n^{-10})$,

$$\left\| \left[\mathcal{P}_{\mathsf{off}\mathsf{-diag}} \left(\overline{\boldsymbol{E}} \, \overline{\boldsymbol{E}}^{\mathsf{T}} \right) \right]^{k} \overline{\boldsymbol{E}} \boldsymbol{V}^{\star} \right\|_{2,\infty} \lesssim \sqrt{\frac{\mu r}{n_{2}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{k} \sqrt{n_{2}} \omega_{\mathsf{max}} \log n$$
(168)

holds for all $0 \le k \le \log n$.

To finish up, note that the union bound tell us that with probability exceeding $1 - O(n^{-10})$,

 $\overline{E} = E.$

This combined with inequality (168) establishes the desired result for the general case.

B.5 Proof of Lemma 3

We first study the case with bounded noise (i.e., the case that (131) always holds). Akin to the proof of Lemma 2, we first intend to show that the following statement holds: for any $0 \le k \le \log n$ and any noise matrix E satisfying Conditions 1 and 2 in Assumption 1 and (131), with probability exceeding $1 - O((n+3)^{2k}n^{-C_2\log n})$ one has

$$\left\| \left[\mathcal{P}_{\mathsf{off}\mathsf{-diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top}\right) \right]^{\ell} \boldsymbol{U}^{\star} \right\|_{2,\infty} \leq C_{3} \sqrt{\frac{\mu r}{n_{1}}} \left(C_{3} \left(\sqrt{n_{1}n_{2}} + n_{1} \right) \omega_{\mathsf{max}}^{2} \log^{2} n \right)^{\ell}$$
(169)

and

$$\left\| \boldsymbol{E}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{\ell} \boldsymbol{U}^{\star} \right\|_{2,\infty} \le C_4 \sqrt{\frac{\mu r}{n_1}} \left(C_3 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n \right)^{\ell} \left(B \log^2 n + \sqrt{n_1} \omega_{\mathsf{max}} \log n \right)$$
(170)

simultaneously for all l obeying $0 \le l \le k$.

Regarding the base case with k = 0, it is self-evident that (169) and (170) hold with probability exceeding $1 - O(n^{-C_2 \log n})$ due to Assumption 1 and (134d). Suppose now that with probability exceeding $1 - O((n + 3)^{2K}n^{-C_2 \log n})$, (169) and (170) hold for all $0 \le \ell \le K$, and we would like to extend the results to k = K + 1. Similar to (140) and (149), one has

$$\begin{aligned} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K} \boldsymbol{U}^{\star} \right\|_{2} \\ &\leq \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right]^{K} \boldsymbol{U}^{\star} \right\|_{2} \\ &+ \sum_{\ell=0}^{K-1} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,:)\top} \right\|_{2}^{2} \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(-i,:)} \boldsymbol{E}^{(-i,:)\top} \right) \right\|^{\ell} \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-1-\ell} \boldsymbol{U}^{\star} \right\|_{2,\infty} \end{aligned}$$
(171)

and

$$\begin{split} & \left\| \boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K+1} \boldsymbol{U}^{\star} \right\|_{2} \\ & \leq \left\| \boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top} \right) \right]^{K+1} \boldsymbol{U}^{\star} \right\|_{2} \\ & + \sum_{\ell=0}^{K} \left\| \boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top} \right) \right]^{\ell} \boldsymbol{E}_{:,j} \right\| \left\| \boldsymbol{E}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-\ell} \boldsymbol{U}^{\star} \right\|_{2,\infty} \end{split}$$

$$+\sum_{\ell=0}^{K} \left\| \boldsymbol{E}_{:,j}^{\top} \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}^{(:,-j)} \boldsymbol{E}^{(:,-j)\top} \right) \right]^{\ell} \mathcal{P}_{\mathsf{diag}} \left(\boldsymbol{E}^{(:,j)} \boldsymbol{E}^{(:,j)\top} \right) \right\|_{1} \\ \cdot \left\| \left[\mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right]^{K-\ell} \boldsymbol{U}^{\star} \right\|_{2,\infty}.$$
(172)

In view of (186), (134d) and Lemma 5, for any E satisfying Conditions 1 and 2 in Assumption 1 and (131), with probability $1 - O(n^{-C_1 \log n})$, for all $i \in [n_1]$, one has

$$\begin{split} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,)\top} \boldsymbol{U}^{\star} \right\|_{2} &\leq C_{5} \left(B \left\| \boldsymbol{E}^{(-i,)\top} \boldsymbol{U}^{\star} \right\|_{2,\infty} \log^{2} n + \omega_{\max} \left\| \boldsymbol{E}^{(-i,)\top} \boldsymbol{U}^{\star} \right\|_{\mathrm{F}} \log n \right) \\ &\leq C_{5} \left(B^{2} \log^{4} n + B \omega_{\mathsf{col}} \log^{3} n \right) \sqrt{\frac{\mu r}{n_{1}}} + C_{5} \sqrt{r} \omega_{\mathsf{max}} \left(B \sqrt{\frac{\mu r}{n_{1}}} \log n + \sqrt{n_{2}} \omega_{\mathsf{max}} \right) \log^{2} n \\ &\leq \left[C_{5} \left(C_{\mathsf{b}}^{2} \sqrt{n_{1} n_{2}} + C_{\mathsf{b}} \sqrt{n_{1} n_{2}} \right) \omega_{\mathsf{max}}^{2} \sqrt{\frac{\mu r}{n_{1}}} + C_{5} \omega_{\mathsf{max}}^{2} \sqrt{\frac{\mu r}{n_{1}}} \left(C_{\mathsf{b}} \sqrt{r n_{2}} + \sqrt{n_{1} n_{2}} \right) \right] \log^{2} n \\ &\leq C_{5} \left(C_{\mathsf{b}} + 1 \right)^{2} \sqrt{n_{1} n_{2}} \omega_{\mathsf{max}}^{2} \sqrt{\frac{\mu r}{n_{1}}} \log^{2} n. \end{split}$$

As a result, with the same probability, we have

$$\left\| \left[\mathcal{P}_{\mathsf{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right] \boldsymbol{U}^{\star} \right\|_{2,\infty} = \max_{1 \le i \le n_1} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,)\top} \boldsymbol{U}^{\star} \right\|_2 \le C_5 \left(C_{\mathsf{b}} + 1 \right)^2 \sqrt{n_1 n_2} \omega_{\mathsf{max}}^2 \sqrt{\frac{\mu r}{n_1}} \log^2 n_1 \left(\frac{1}{2} \sum_{i \le n_1} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,)\top} \boldsymbol{U}^{\star} \right\|_2 \right) \right\|_2} \le C_5 \left(C_{\mathsf{b}} + 1 \right)^2 \sqrt{n_1 n_2} \omega_{\mathsf{max}}^2 \sqrt{\frac{\mu r}{n_1}} \log^2 n_2 \left(\frac{1}{2} \sum_{i \le n_1} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,)\top} \boldsymbol{U}^{\star} \right\|_2 \right) \right\|_2} \le C_5 \left(\frac{1}{2} \sum_{i \le n_1} \left\| \boldsymbol{E}_{i,:} \boldsymbol{E}^{(-i,)\top} \boldsymbol{U}^{\star} \right\|_2 \right)$$

and

$$\left\| \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right] \boldsymbol{U}^{\star} \right\|_{\mathsf{F}} \leq \sqrt{n_{1}} \left\| \left[\mathcal{P}_{\mathsf{off}\mathsf{-}\mathsf{diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right] \boldsymbol{U}^{\star} \right\|_{2,\infty} \leq C_{5} \left(C_{\mathsf{b}} + 1 \right)^{2} \sqrt{n_{1}n_{2}} \omega_{\mathsf{max}}^{2} \sqrt{\mu r} \log^{2} n.$$

Equipped with the previous two inequalities, we can carry out the induction step using a similar argument of Lemma 2.

For the general case where the noise matrix E satisfies Assumption 2, one can get the desired result by using the same truncation trick as in Section B.4.2.

B.6 Proof of Lemma 4

Bounding the spectrum of Σ . Let us first develop an upper bound (resp. lower bound) on the singular value perturbation $|\tilde{\sigma}_i - \sigma_i^*|$ (resp. the spectral gap $\tilde{\sigma}_{r'}^2 - \tilde{\sigma}_{r'+1}^2$ for any $r' \in \mathcal{R}'$ defined in (87)). Weyl's inequality tell us that, for all $1 \leq i \leq r$,

$$\begin{split} |\widetilde{\sigma}_i - \sigma_i^{\star}| &\leq \|\boldsymbol{E}\boldsymbol{V}^{\star}\| \lesssim B\sqrt{\frac{\mu r}{n_2}}\log^2 n + \left(r\omega_{\max}^2 + n_1\omega_{\max}^2\right)^{1/2}\log n\\ &\lesssim \sqrt{\frac{\mu r}{n_2}}\omega_{\max}\frac{\sqrt{n_2}}{\log n}\log^2 n + \sqrt{n_1}\omega_{\max}\log n\\ &\leq \sqrt{C_5}\sqrt{n_1}\omega_{\max}\log n \leq \frac{\sigma_r^{\star}}{40r} \end{split}$$

holds with probability at least $1 - O(n^{-10})$ for some constant $C_5 > 0$. Here, the first line invokes Lemma 5, the second line relies on Assumption 2, and the last line makes use of the assumption (20b). Consequently,

$$\widetilde{\sigma}_{r'} - \widetilde{\sigma}_{r'+1} \ge \sigma_{r'}^{\star} - \sigma_{r'+1}^{\star} - \frac{\sigma_r^{\star}}{20r} \ge \frac{4\left(\sigma_{r'}^{\star} - \sigma_{r'+1}^{\star}\right)}{5},$$

where we have made use of the definition of \mathcal{R}' in (87) and the fact that $\sigma_{r'}^{\star} - \sigma_{r'+1}^{\star} = \frac{\sigma_{r'}^{\star 2} - \sigma_{r'+1}^{\star 2}}{\sigma_{r'}^{\star} + \sigma_{r'+1}^{\star}} \ge \frac{\sigma_{r'}^{\star 2} - \sigma_{r'+1}^{\star 2}}{2\sigma_{r'}^{\star}}$. This further gives

$$\widetilde{\sigma}_{r'}^2 - \widetilde{\sigma}_{r'+1}^2 = (\widetilde{\sigma}_{r'} - \widetilde{\sigma}_{r'+1}) (\widetilde{\sigma}_{r'} + \widetilde{\sigma}_{r'+1}) \ge \frac{4 \left(\sigma_{r'}^{\star} - \sigma_{r'+1}^{\star}\right)}{5} \left(\sigma_{r'}^{\star} + \sigma_{r'+1}^{\star} - \frac{\sigma_{r}^{\star}}{5r}\right) \ge \frac{1}{2} \left(\sigma_{r'}^{\star 2} - \sigma_{r'+1}^{\star 2}\right).$$

Bounding the noise size $\|\mathcal{P}_{\mathsf{off}\mathsf{-diag}}(\boldsymbol{E}\boldsymbol{E}^{\top} - \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top})\|$. We now move on to control $\|\mathcal{P}_{\mathsf{off}\mathsf{-diag}}(\boldsymbol{E}\boldsymbol{E}^{\top} - \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top})\|$. Towards this end, Lemma 7 tells us that, with probability at least $1 - O(n^{-10})$,

$$\begin{aligned} \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right\| &\lesssim B^2 \log^4 n + \sqrt{n_1} \omega_{\mathsf{max}} \left(\sqrt{n_1} + \sqrt{n_2} \right) \omega_{\mathsf{max}} \log^2 n \\ &\lesssim \left(\frac{(n_1 n_2)^{1/4}}{\log n} \omega_{\mathsf{max}} \right)^2 \log^4 n + \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n \\ &\asymp \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n, \end{aligned}$$

$$(173)$$

where the second line results from Assumption 2. In view of (88a) and (173), with probability exceeding $1 - O(n^{-10})$, we have

$$\begin{split} \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} - \boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top} \right) \right\| &\leq \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right\| + \left\| \mathcal{P}_{\mathsf{off}\text{-}\mathsf{diag}} \left(\boldsymbol{E}\boldsymbol{V}^{\star}\boldsymbol{V}^{\star\top}\boldsymbol{E}^{\top} \right) \right\| \\ &\leq C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n + 2 \left\| \boldsymbol{E}\boldsymbol{V}^{\star} \right\|^2 \\ &\leq C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n + 2 C_5 n_1 \omega_{\mathsf{max}}^2 \log^2 n \\ &\leq 3 C_5 \left(\sqrt{n_1 n_2} + n_1 \right) \omega_{\mathsf{max}}^2 \log^2 n \end{split}$$

for some large enough constant $C_5 > 0$.

Bounding the incoherence concerning $\|\tilde{U}\|_{2,\infty}$. We now turn to the incoherence property w.r.t. \tilde{U} . Lemma 5 together with Assumption 2 reveals that with probability exceeding $1 - O(n^{-10})$,

$$\begin{aligned} \left\| \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\|_{2,\infty} &\leq \left\| \boldsymbol{U}^{\star} \right\|_{2,\infty} \left\| \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \leq \sqrt{\frac{\mu r}{n_1}} \left\| \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \\ &\lesssim \sqrt{\frac{\mu r}{n_1}} \left(B \frac{\mu r}{\sqrt{n_1 n_2}} \log^2 n + \sqrt{r} \omega_{\max} \log n \right) \\ &\lesssim \sqrt{\frac{\mu r}{n_1}} \left(\frac{\sqrt{n_2}}{\log n} \frac{\mu r}{\sqrt{n_1 n_2}} \omega_{\max} \log^2 n + \sqrt{r} \omega_{\max} \log n \right) \\ &\lesssim \sqrt{\frac{\mu r}{n_1}} \sqrt{\mu r} \omega_{\max} \log n, \end{aligned}$$
(174)

where the first line follows from Definition 1, the third line makes use of Assumption 2, and the last line holds due to the assumption $\mu r^3 \leq n_1$. Putting (55), (88a) and (174) together, we can demonstrate that with probability exceeding $1 - O(n^{-10})$,

$$\begin{split} \left\| \boldsymbol{U}^{\star} \boldsymbol{U}^{\star \top} \widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}} \right\|_{2,\infty} &\leq C_5 \left(\sqrt{\mu r} \omega_{\max} \log n + \sqrt{\frac{\mu r}{n_1}} \sqrt{\mu r} \omega_{\max} \log n \right) \frac{2}{\sigma_r^{\star}} \\ &\leq \frac{4C_5 \sqrt{\mu r} \omega_{\max} \log n}{\sigma_r^{\star}} \leq \sqrt{\frac{\mu r}{n_1}}, \end{split}$$

where the last inequality follows from the assumptions (20). This in turn indicates that

$$\left\|\widetilde{\boldsymbol{U}}\right\|_{2,\infty} \leq \left\|\boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}}\right\|_{2,\infty} + \left\|\boldsymbol{U}^{\star}\boldsymbol{U}^{\star\top}\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}}\right\|_{2,\infty} \leq 2\sqrt{\frac{\mu r}{n_1}}.$$

C Proofs for corollaries

C.1 Proof of Corollary 1

First, by virtue of the standard tail bound of sub-Gaussian random variables (cf. Vershynin (2010, Lemma 5.5)), we can easily verify that Assumption 1 holds with the following parameters:

$$\omega_{\max} = \omega$$
 and $B = C_{\mathsf{B}}\omega\log(n+d) \lesssim \omega \frac{\min\left\{(nd)^{1/4}, n^{1/2}\right\}}{\log(n+d)}$

for some constant $C_{\mathsf{B}} > 0$.

Next, let us look at several properties of the matrix $\boldsymbol{X} = [\boldsymbol{x}_1 \ \dots \ \boldsymbol{x}_n] \in \mathbb{R}^{d \times n}$. It is seen that

$$oldsymbol{X} = oldsymbol{U}^\star oldsymbol{\Lambda}^{\star 1/2} oldsymbol{F}^\star, \qquad ext{with } oldsymbol{F}^\star = [oldsymbol{f}_1, \dots, oldsymbol{f}_n] \in \mathbb{R}^{r imes n},$$

where $F_{i,j}^{\star} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ for all $(i,j) \in [r] \times [n]$. In view of Vershynin (2010, Corollary 5.35), we know that with probability exceeding $1 - O((n+d)^{-10})$,

$$\sqrt{n}/2 \le \sqrt{n} - \sqrt{r} - \sqrt{20\log(n+d)} \le \sigma_r \left(\boldsymbol{F}^\star \right) \le \sigma_1 \left(\boldsymbol{F}^\star \right) \le \sqrt{n} + \sqrt{r} + \sqrt{20\log(n+d)} \le 2\sqrt{n}.$$
(175)

By the min-max principle for singular values, for all $1 \le i \le r$, one has

$$\lambda_{i}^{\star 1/2} \sigma_{r} \left(\boldsymbol{F}^{\star} \right) = \min_{\substack{\boldsymbol{S}: \dim(\boldsymbol{S}) = r - i + 1 \ \boldsymbol{x} \in \boldsymbol{S}, \|\boldsymbol{x}\|_{2} = 1}} \max \left\| \boldsymbol{x}^{\top} \boldsymbol{\Lambda}^{\star 1/2} \right\| \sigma_{r} \left(\boldsymbol{F}^{\star} \right)$$

$$\leq \sigma_{i} \left(\boldsymbol{X}^{\star} \right) = \sigma_{i} \left(\boldsymbol{\Lambda}^{\star 1/2} \boldsymbol{F}^{\star} \right)$$

$$= \min_{\substack{\boldsymbol{S}: \dim(\boldsymbol{S}) = r - i + 1 \ \boldsymbol{x} \in \boldsymbol{S}, \|\boldsymbol{x}\|_{2} = 1}} \max \left\| \boldsymbol{x}^{\top} \boldsymbol{\Lambda}^{\star 1/2} \boldsymbol{F}^{\star} \right\|$$

$$\leq \min_{\substack{\boldsymbol{S}: \dim(\boldsymbol{S}) = r - i + 1 \ \boldsymbol{x} \in \boldsymbol{S}, \|\boldsymbol{x}\|_{2} = 1}} \| \boldsymbol{x}^{\top} \boldsymbol{\Lambda}^{\star 1/2} \| \| \boldsymbol{F}^{\star} \|$$

$$= \lambda_{i}^{\star 1/2} \sigma_{1} \left(\boldsymbol{F}^{\star} \right). \tag{176}$$

Therefore, with probability exceeding $1 - O((n+d)^{-10})$, we obtain

$$\sigma_i(\mathbf{X}^\star) \asymp \sqrt{n\lambda_i^\star} \qquad \text{for all } 1 \le i \le r.$$
(177)

In fact, the relation (176) taken together with (175) and (34a) yields a more concrete lower bound

$$\sigma_i\left(\boldsymbol{X}^{\star}\right) \ge \sqrt{n\lambda_i^{\star}}/2 \ge C_0 r\left[\left(dn\right)^{1/4} + d^{1/2}\right] \log(n+d) \qquad \text{for all } 1 \le i \le r$$

Hence, the signal-to-noise ratio condition in Theorem 2 is satisfied (where we take $n_1 = d$ and $n_2 = n$). Additionally, letting $V^* \in \mathcal{O}^{n,r}$ denote the right singular space of X^* , we see from the proof of (Cai et al., 2021, Corollary 2) that with probability exceeding $1 - O((n+d)^{-10})$,

$$\|\boldsymbol{V}^{\star}\|_{2,\infty} \leq \sqrt{\frac{C_2 r \log(n+d)}{n}}$$

for some constant $C_2 > 0$. Consequently, we have

$$\mu \le \mu_{\mathsf{pc}} \lor C_2 \log(n+d) \lesssim \frac{d}{r^3},$$

where μ_{pc} is defined in (33) and the last inequality arises from the assumption (34b).

Now, we see that with probability at least $1 - O((n+d)^{-10})$, all conditions in Theorem 2 are satisfied. Thus, apply Theorem 2 and (177) to yield that: with probability exceeding $1 - O((n+d)^{-10})$,

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}} - \boldsymbol{U}^{\star}\| \lesssim \frac{\sqrt{dn}\,\omega^2\log^2(n+d)}{n\lambda_r^{\star}} + \frac{\sqrt{d}\,\omega\log(n+d)}{\sqrt{n\lambda_r^{\star}}} \asymp \frac{\sqrt{d/n}\,\omega^2\log^2(n+d)}{\lambda_r^{\star}} + \frac{\sqrt{d/n}\,\omega\log(n+d)}{\sqrt{\lambda_r^{\star}}}$$

and

$$\|\boldsymbol{U}\boldsymbol{R}_{\boldsymbol{U}}-\boldsymbol{U}^{\star}\|_{2,\infty} \lesssim \sqrt{\frac{\mu_{\mathsf{pc}}+\log(n+d)}{d}} \left(\frac{\sqrt{d/n}\,\omega^2\log^2(n+d)}{\lambda_r^{\star}} + \frac{\sqrt{d/n}\,\omega\log(n+d)}{\sqrt{\lambda_r^{\star}}}\right),$$

provided that the number of iterations satisfy (35a)-(35b).

C.2 Proof of Corollary 2

For notational convenience, we let $\mathbf{Y}_i \in \mathbb{R}^{n_i \times (n_1 n_2 n_3/n_i)}$ (resp. \mathbf{X}_i^{\star} and \mathbf{E}_i) denote the *i*-th matricization of $\boldsymbol{\mathcal{Y}}$ (resp. $\boldsymbol{\mathcal{X}}^{\star}$ and $\boldsymbol{\mathcal{E}}$). We need to check that all assumptions in Theorem 2 are satisfied for the *i*-th matricization.

Firstly, it can be easily verifid that Assumption 2 holds for E_i with $\omega_{max} = \omega$ and $B \simeq \omega \log n$. In addition, taking the assumption $n_1 \simeq n_2 \simeq n_3$ and (39a) together imply that

$$\frac{\sigma_{i,r_i}^{\star}}{\omega} \ge C_0 r \left[(n_1 n_2 n_3)^{1/4} + n_i^{1/2} \right] \log \left(n_i \lor (n_1 n_2 n_3/n_i) \right)$$

for some large enough constant $C_0 > 0$, thus justifying the SNR condition (20a) in Theorem 2. Next, let $V_i^* \in \mathcal{O}^{n_1 n_2 n_3/n_i, r_i}$ denote the right singular space of X_i^* and define

$$\mu(\mathbf{X}_{i}^{\star}) = \max\left\{\frac{n_{i}}{r_{i}} \|\mathbf{U}_{i}^{\star}\|_{2,\infty}^{2}, \frac{n_{1}n_{2}n_{3}/n_{i}}{r_{i}} \|\mathbf{V}_{i}^{\star}\|_{2,\infty}^{2}\right\}$$

Given that $X_1^{\star} = U_1^{\star} \mathcal{M}_1(\mathcal{S}^{\star}) (U_3^{\star} \otimes U_2^{\star})^{\top}$, we can invoke (38) and (39b) to obtain

$$\|\boldsymbol{V}_{1}^{\star}\|_{2,\infty} \leq \|\boldsymbol{U}_{3}^{\star} \otimes \boldsymbol{U}_{2}^{\star}\|_{2,\infty} \leq \|\boldsymbol{U}_{2}^{\star}\|_{2,\infty} \|\boldsymbol{U}_{3}^{\star}\|_{2,\infty} \leq \sqrt{\frac{\mu^{2} r_{2} r_{3}}{n_{2} n_{3}}}$$

and

$$\mu\left(\boldsymbol{X}_{1}^{\star}\right) \leq \max\left\{\mu, \mu^{2} \frac{r_{2} r_{3}}{r_{1}}\right\} \lesssim \frac{n_{1}}{r_{1}^{3}}.$$

Therefore, all conditions and assumptions in Theorem 2 are satisfied. Consequently, invoke Theorem 2 to show that, with probability exceeding $1 - O(n^{-10})$,

$$\begin{split} \left\| \widehat{\boldsymbol{U}}_{1}^{0} \boldsymbol{R}_{\widehat{\boldsymbol{U}}_{1}^{0}} - \boldsymbol{U}_{1}^{\star} \right\|_{2,\infty} \lesssim \sqrt{\frac{\mu\left(\boldsymbol{X}_{1}^{\star}\right) r_{1}}{n_{1}}} \left(\frac{\sqrt{n_{1}n_{2}n_{3}}\omega^{2}\log^{2}n}{\sigma_{1,r_{1}}^{\star2}} + \frac{\sqrt{n_{1}}\omega\log n}{\sigma_{1,r_{1}}^{\star}} \right) \\ \leq \frac{\mu r}{\sqrt{n_{1}}} \left(\frac{\sqrt{n_{1}n_{2}n_{3}}\omega^{2}\log^{2}n}{\sigma_{\min}^{\star2}} + \frac{\sqrt{n_{1}}\omega\log n}{\sigma_{\min}^{\star}} \right) \end{split}$$

and

$$\left\|\widehat{\boldsymbol{U}}_{1}^{0}\boldsymbol{R}_{\widehat{\boldsymbol{U}}_{1}^{0}}-\boldsymbol{U}^{\star}\right\|\lesssim\frac{n^{3/2}\omega^{2}\log^{2}n}{\sigma_{\min}^{\star2}}+\frac{\sqrt{n}\omega\log n}{\sigma_{\min}^{\star}}.$$

Similarly, one can show that with probability at least $1 - O(n^{-10})$, (41a) and (41b) holds for i = 2 and 3, thereby establishing the first part of Corollary 2.

When it comes to the second part, we can directly use the same argument in the proof of Zhang and Xia (2018, Theorem 1) if the following two claims are valid with probability exceeding $1 - O(n^{-10})$:

$$\max_{\boldsymbol{V}_{i}\in\mathbb{R}^{n_{i}\times r_{i}},\|\boldsymbol{V}_{i}\|\leq1}\max\left\{\left\|\boldsymbol{E}_{1}\left(\boldsymbol{V}_{3}\otimes\boldsymbol{V}_{2}\right)\right\|,\left\|\boldsymbol{E}_{2}\left(\boldsymbol{V}_{1}\otimes\boldsymbol{V}_{3}\right)\right\|,\left\|\boldsymbol{E}_{3}\left(\boldsymbol{V}_{2}\otimes\boldsymbol{V}_{1}\right)\right\|\right\}\lesssim\sqrt{nr},$$
(178)

and

$$\max\left\{\left\|\boldsymbol{E}_{1}\left(\boldsymbol{U}_{3}^{\star}\otimes\boldsymbol{U}_{2}^{\star}\right)\right\|,\left\|\boldsymbol{E}_{2}\left(\boldsymbol{U}_{1}^{\star}\otimes\boldsymbol{U}_{3}^{\star}\right)\right\|,\left\|\boldsymbol{E}_{3}\left(\boldsymbol{U}_{2}^{\star}\otimes\boldsymbol{U}_{1}^{\star}\right)\right\|\right\}\lesssim\sqrt{n}.$$
(179)

In fact, (179) is a direct consequence of Zhou et al. (2022, Lemma A.2) (or Lemma 8.2 in its arxiv version) with

$$A = I_{n_1} \text{ (resp. } I_{n_2} \text{ and } I_{n_3} \text{)} \text{ and } B = U_3^{\star} \otimes U_2^{\star} \text{ (resp. } U_1^{\star} \otimes U_3^{\star} \text{ and } U_2^{\star} \otimes U_1^{\star} \text{)}$$

whereas (178) can be proved by combining Zhou et al. (2022, Lemma A.2) and the standard epsilon-net argument in the proof of Zhang and Xia (2018, Lemma 5). We omit the details here for the sake of brevity.

D Technical lemmas

In this section, we collect a couple of useful technical lemmas and provide proofs. Before continuing, we note that Assumption 1 and 2 are subsumed as special cases of the following assumption:

Assumption 3. Suppose that the noise components $\{E_{i,j}\}$ satisfy the following conditions:

- 1. The $E_{i,j}$'s are statistically independent and zero-mean;
- 2. $\operatorname{Var}[E_{i,j}] = \omega_{i,j}^2 \leq \omega_{\max}^2$ for all $(i,j) \in [n_1] \times [n_2]$;
- 3. For any $(i, j) \in [n_1] \times [n_2]$, one has $\mathbb{P}(|E_{i,j}| > B) \leq \varepsilon$ for some quantity B, where ε is some quantity within $[0, C_b n^{-10}]$ for some universal constant $C_b > 0$.

Let us begin with several tail bounds regarding the spectral norm of linear functions of $\boldsymbol{E} = [E_{i,j}]_{(i,j)\in[n_1]\times[n_2]}$.

Lemma 5. Suppose that Assumption 3 holds. Then there exists some large (resp. small) enough constant $C_1 > 0$ ($c_1 > 0$) such that for any $x \ge C_1 \sqrt{\log n}$, with probability exceeding $1 - O(e^{-c_1 x^2}) - n_1 n_2 \varepsilon$ one has

$$\|\boldsymbol{E}\boldsymbol{V}^{\star}\| \lesssim B\sqrt{\frac{\mu_2 r}{n_2}}x^2 + \left(\left(\frac{\mu r}{n_2}\omega_{\mathsf{row}}^2 \wedge r\omega_{\mathsf{max}}^2\right) + \omega_{\mathsf{col}}^2\right)^{1/2}x,\tag{180a}$$

$$\left\| \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \lesssim B \frac{\mu r}{\sqrt{n_1 n_2}} x^2 + \left[\left(\sqrt{\frac{\mu r}{n_2}} \omega_{\mathsf{row}} + \sqrt{\frac{\mu r}{n_1}} \omega_{\mathsf{col}} \right) \wedge \sqrt{r} \omega_{\mathsf{max}} \right] x, \tag{180b}$$

$$\|\boldsymbol{E}\boldsymbol{V}^{\star}\|_{2,\infty} \lesssim \left(Bx^2 + \omega_{\mathsf{row}}x\right)\sqrt{\frac{\mu_2 r}{n_2}},\tag{180c}$$

$$\|\boldsymbol{E}\| \lesssim Bx + (\omega_{\mathsf{row}} + \omega_{\mathsf{col}}) \,. \tag{180d}$$

Proof of Lemma 5. We start with the case $\varepsilon = 0$, i.e., $|E_{i,j}| \leq B$ holds deterministically (see Assumption 3).

• First, express EV^{\star} as a sum of zero-mean independent random matrices as follows

$$\boldsymbol{E}\boldsymbol{V}^{\star} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E_{i,j} \boldsymbol{e}_i \boldsymbol{V}_{j,:}^{\star}$$

From the definition (7) and the incoherence condition in Definition 1, one can verify that

$$L_{1} := \max_{1 \le i \le n_{1}, 1 \le j \le n_{2}} \left\| E_{i,j} \boldsymbol{e}_{i} \boldsymbol{V}_{j,:}^{\star} \right\| \le B \sqrt{\frac{\mu_{2} r}{n_{2}}}$$

and

$$V_{1} := \max\left\{ \left\| \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} \mathbb{E}\left[E_{i,j}^{2} \right] \left\| \boldsymbol{V}_{j,:}^{\star} \right\|_{2}^{2} \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\top} \right\|, \left\| \sum_{j=1}^{n_{2}} \sum_{i=1}^{n_{1}} \mathbb{E}\left[E_{i,j}^{2} \right] \boldsymbol{V}_{j,:}^{\star \top} \boldsymbol{V}_{j,:}^{\star} \right\| \right\}$$
$$\leq \left(\frac{\mu_{2} r}{n_{2}} \omega_{\text{row}}^{2} \wedge r \omega_{\text{max}}^{2} \right) + \omega_{\text{col}}^{2},$$

where the last line also uses the facts that $\sum_{j} \|\boldsymbol{V}_{j,:}^{\star}\|_{2}^{2} = r$ and $\sum_{j} \boldsymbol{V}_{j,:}^{\star\top} \boldsymbol{V}_{j,:}^{\star} = \boldsymbol{V}^{\star\top} \boldsymbol{V}^{\star} = \boldsymbol{I}_{r}$. Applying the matrix Bernstein inequality (Tropp et al., 2015) leads to, with probability exceeding $1 - O(e^{-c_{1}x^{2}})$,

$$\|\boldsymbol{E}\boldsymbol{V}^{\star}\| \lesssim L_{1}x^{2} + \sqrt{V_{1}}x \lesssim B\sqrt{\frac{\mu_{2}r}{n_{2}}}x^{2} + \sqrt{\left(\left(\frac{\mu_{2}r}{n_{2}}\omega_{\mathsf{row}}^{2} \wedge r\omega_{\mathsf{max}}^{2}\right) + \omega_{\mathsf{col}}^{2}\right)}x$$

for any $x \ge C_1 \sqrt{\log n}$, where $c_1, C_1 > 0$ are some suitable numerical constants.

• When it comes to $U^{\star \top} E V^{\star}$, we decompose it into the following zero-mean and independent terms:

$$\boldsymbol{U}^{\star\top}\boldsymbol{E}\boldsymbol{V}^{\star} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E_{i,j} \boldsymbol{U}_{i,:}^{\top} \boldsymbol{V}_{j,:}^{\star}$$

Similar to the above arguments, it follows from (7) and Definition 1 that

$$L_{2} := \max_{1 \le i \le n_{1}, 1 \le j \le n_{2}} \left\| E_{i,j} \boldsymbol{U}_{i,:}^{\top} \boldsymbol{V}_{j,:}^{\star} \right\| \le B \sqrt{\frac{\mu r}{n_{1}}} \sqrt{\frac{\mu r}{n_{2}}} = B \frac{\mu r}{\sqrt{n_{1} n_{2}}}$$

and

$$V_{2} := \max\left\{ \left\| \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} \mathbb{E}\left[E_{i,j}^{2}\right] \left\| \mathbf{V}_{j,:}^{\star} \right\|_{2}^{2} \mathbf{U}_{i,:}^{\top} \mathbf{U}_{i,:} \right\|, \left\| \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} \mathbb{E}\left[E_{i,j}^{2}\right] \left\| \mathbf{U}_{i,:} \right\|_{2}^{2} \mathbf{V}_{j,:}^{\star \top} \mathbf{V}_{j,:}^{\star} \right\| \right\}$$
$$\leq \left[\frac{\mu r}{n_{2}} \omega_{\mathsf{row}}^{2} + \frac{\mu r}{n_{1}} \omega_{\mathsf{col}}^{2} \right] \wedge r \omega_{\mathsf{max}}^{2}.$$

The matrix Bernstein inequality reveals that with probability exceeding $1 - O(e^{-c_1 x^2})$,

$$\left\| \boldsymbol{U}^{\star \top} \boldsymbol{E} \boldsymbol{V}^{\star} \right\| \lesssim L_2 x^2 + \sqrt{V_2} x \lesssim B \frac{\mu r}{\sqrt{n_1 n_2}} x^2 + \left[\left(\sqrt{\frac{\mu r}{n_2}} \omega_{\mathsf{row}} + \sqrt{\frac{\mu r}{n_1}} \omega_{\mathsf{col}} \right) \wedge \sqrt{r} \omega_{\mathsf{max}} \right] x.$$

• Additionally, (180c) and (180d) are direct consequences of Cai et al. (2021, Lemma 12) and Chen et al. (2021b, Theorem 3.4), respectively.

We now move on to the more general case with $\varepsilon > 0$ (see Assumption 3). Denoting by $\widetilde{E}_{i,j}$ the centered truncated noise as follows:

$$\widetilde{E}_{i,j} = E_{i,j} \mathbb{1}_{\{|E_{i,j}| \le B\}} - \mathbb{E} \left[E_{i,j} \mathbb{1}_{\{|E_{i,j}| \le B\}} \right].$$
(181)

we see that

$$\mathsf{Var}\big(\widetilde{E}_{i,j}\big) \leq \mathbb{E}\left[E_{i,j}^2 \mathbb{1}_{\{|E_{i,j}| \leq B\}}\right] \leq \mathbb{E}\left[E_{i,j}^2\right] = \omega_{i,j}^2$$

and

$$\left|\widetilde{E}_{i,j}\right| \le B + B = 2B.$$

The previous argument shows that with probability exceeding $1 - O(e^{-c_1 x^2})$,

inequalities
$$(180a) - (180d)$$
 hold if we replace E with \tilde{E} . (182)

Next, let \overline{E} denote the matrix with the (i, j)-th entry $\overline{E}_{i,j} = E_{i,j} \mathbb{1}_{\{|E_{i,j}| \leq B\}}$ for all $(i, j) \in [n_1] \times [n_2]$. In view of the Cauchy-Schwarz inequality and the assumption $\mathbb{E}[E_{i,j}] = 0$, one has

$$\left|\mathbb{E}\left[\overline{E}_{i,j}\right]\right| = \left|\mathbb{E}\left[E_{i,j}\right] - \mathbb{E}\left[E_{i,j}\mathbb{1}_{\{|E_{i,j}|>B\}}\right]\right| = \left|\mathbb{E}\left[E_{i,j}\mathbb{1}_{\{|E_{i,j}|>B\}}\right]\right| \le \left(\mathbb{E}\left[E_{i,j}^{2}\right]\mathbb{E}\left[\mathbb{1}_{\{|E_{i,j}|>B\}}\right]\right)^{1/2} \le \omega_{i,j}\sqrt{\varepsilon},$$

and as a result,

$$\left\|\mathbb{E}\left[\overline{E}\right]\right\| \le \left\|\mathbb{E}\left[\overline{E}\right]\right\|_{\mathrm{F}} \le \sqrt{n_1 n_2} \max_{i,j} \left|\mathbb{E}\left[\overline{E}_{i,j}\right]\right| \le \omega_{\max} \sqrt{n_1 n_2 \varepsilon} \lesssim \frac{\omega_{\max}}{n^4}.$$
(183)

Assumption 3 and the union bound tell us that with probability at least $1 - n_1 n_2 \varepsilon$, for all $i, j \in [n_1] \times [n_2]$,

$$E_{i,j} = E_{i,j} \mathbb{1}_{\{|E_{i,j}| \le B\}},$$

which means

$$E = \overline{E}$$

This combined with (183) yields that with probability exceeding $1 - n_1 n_2 \varepsilon$,

$$\left\| \boldsymbol{E} - \widetilde{\boldsymbol{E}} \right\| = \left\| \mathbb{E} \left[\overline{\boldsymbol{E}} \right] \right\| \lesssim \frac{\omega_{\max}}{n^4}.$$
 (184)

On the event $\mathcal{E}_1 = \{(182) \text{ and } (184) \text{ hold}\}$, we can apply the triangle inequality to show that

$$\begin{split} \| \boldsymbol{E} \boldsymbol{V}^{\star} \| &\leq \left\| \widetilde{\boldsymbol{E}} \boldsymbol{V}^{\star} \right\| + \left\| \left(\boldsymbol{E} - \widetilde{\boldsymbol{E}} \right) \boldsymbol{V}^{\star} \right\| \\ &\lesssim B \sqrt{\frac{\mu_2 r}{n_2}} x^2 + \left(\left(\frac{\mu_2 r}{n_2} \omega_{\mathsf{row}}^2 \wedge r \omega_{\mathsf{max}}^2 \right) + \omega_{\mathsf{col}}^2 \right)^{1/2} x + \left\| \boldsymbol{E} - \widetilde{\boldsymbol{E}} \right\| \\ &\leq B \sqrt{\frac{\mu_2 r}{n_2}} x^2 + \left(\left(\frac{\mu_2 r}{n_2} \omega_{\mathsf{row}}^2 \wedge r \omega_{\mathsf{max}}^2 \right) + \omega_{\mathsf{col}}^2 \right)^{1/2} x \end{split}$$

for any $x \ge C_1 \sqrt{\log n}$. Similarly, one can show that on the same event, (180b)-(180d) hold.

Next, we provide a few more tail bounds concerning the $\ell_{2,\infty}$ norm and sum of squares concerning E.

Lemma 6. Suppose that Assumption 3 holds. There exists some sufficiently large constant $C_2 > 0$ such that for any fixed matrix \mathbf{W}_1 and \mathbf{W}_2 , with probability exceeding $1 - O(n^{-C_2 \log n}) - n_1 n_2 \varepsilon$ one has

$$\|\boldsymbol{E}\boldsymbol{W}_{1}\|_{2,\infty} \lesssim B \|\boldsymbol{W}_{1}\|_{2,\infty} \log^{2} n + \omega_{\max} \|\boldsymbol{W}_{1}\|_{\mathrm{F}} \log n,$$
(185a)

$$\max_{i \in [n_1]} \sum_{j \in [n_2]} E_{i,j}^2 \lesssim B^2 \log^2 n + \omega_{\mathsf{row}}^2, \tag{185b}$$

$$\max_{j \in [n_2]} \sum_{i \in [n_1]} E_{i,j}^2 \lesssim B^2 \log^2 n + \omega_{\mathsf{col}}^2, \tag{185c}$$

$$\max_{j \in [n_2]} \left\| \left(\boldsymbol{E}_{:,j} \right)^\top \boldsymbol{W}_2 \right\|_2 \lesssim \left(B \log^2 n + \omega_{\mathsf{col}} \log n \right) \left\| \boldsymbol{W}_2 \right\|_{2,\infty}.$$
(185d)

Proof of Lemma 6. We again consider the case $\varepsilon = 0$ first (see Assumption 3). In this case, (185b)-(185d) are basically direct consequences of (Cai et al., 2021, Lemma 12). The only difference is we require a higher probability here $(1 - O(n^{-C_2 \log n}))$ instead of $1 - O(n^{-20})$, which leads to an extra log *n* factor in our bounds. Turning to (185a), we note that for any $i \in [n_1]$, $\mathbf{E}_{i,:}\mathbf{W} = \sum_{j \in [n_2]} E_{i,j}\mathbf{W}_{j,:}$ is a sum of n_2 independent zero-mean vectors. In light of the following key quantities:

$$L := \max_{j \in [n_2]} \|E_{i,j} W_{j,:}\|_2 \le B \|W\|_{2,\infty}$$

and

$$V := \sum_{j \in [n_2]} \mathbb{E} \left[E_{i,j}^2 \right] \left\| \mathbf{W}_{j,:} \right\|_2^2 \le \omega_{\max}^2 \sum_{j \in [n_2]} \left\| \mathbf{W}_{j,:} \right\|_2^2 = \omega_{\max}^2 \left\| \mathbf{W} \right\|_{\mathrm{F}}^2,$$

we can apply the matrix Bernstein inequality to show that: with probability exceeding $1 - n^{-C_3 \log n}$,

$$\|\boldsymbol{E}_{i,:}\boldsymbol{W}\|_{2} \lesssim L\log^{2} n + \sqrt{V}\log n \lesssim B \|\boldsymbol{W}\|_{2,\infty}\log^{2} n + \omega_{\max} \|\boldsymbol{W}\|_{F}\log n$$
(186)

holds for some numerical constant $C_3 > 0$. The union bound then shows that with probability exceeding $1 - n \cdot n^{-C_3 \log n} \ge 1 - n^{-C_2 \log n}$ (for some numerical constant $C_2 > 0$),

$$\left\|\boldsymbol{E}\boldsymbol{W}\right\|_{2,\infty} = \max_{i \in [n_1]} \left\|\boldsymbol{E}_{i,:}\boldsymbol{W}\right\|_2 \lesssim B \left\|\boldsymbol{W}\right\|_{2,\infty} \log^2 n + \omega_{\max} \left\|\boldsymbol{W}\right\|_{\mathrm{F}} \log n.$$

When it comes to the more general case with $\varepsilon > 0$, repeating a similar argument as in the proof of Lemma 5 immediately helps us finish the proof of Lemma 5.

The next lemma gathers a spectral norm upper bound on the Gram matrix EE^{\top} after diagonal deletion.

Lemma 7. Assume that Assumption 3 holds. Then there exists some large (resp. small) constant $C_1 > 0$ $(c_1 > 0)$ such that: for any $x \ge C_1 \sqrt{\log n}$, with probability exceeding $1 - O(e^{-c_1 x^2}) - n_1 n_2 \varepsilon$ one has

$$\left\| \mathcal{P}_{\mathsf{off}\operatorname{-diag}}\left(\boldsymbol{E}\boldsymbol{E}^{\top} \right) \right\| \lesssim B^2 x^4 + \omega_{\mathsf{col}}\left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}} \right) x^2.$$

Proof of Lemma 7. In view of Cai et al. (2021, Section B.2.1) (or more precisely, we use the proof therein but change the probability slightly), we know that with probability $1 - O(e^{-c_1 x^2})$,

$$\left\| \mathcal{P}_{\mathsf{off-diag}} \left(\widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top} \right) \right\| \lesssim B^2 x^4 + \omega_{\mathsf{col}} \left(\omega_{\mathsf{row}} + \omega_{\mathsf{col}} \right) x^2, \tag{187}$$

where \tilde{E} is defined in (181). Let \mathcal{E}_2 denote the following event:

$$\mathcal{E}_2 := \left\{ (184) \text{ and } (187) \text{ hold, and } \left\| \widetilde{E} \right\| \lesssim Bx + (\omega_{\mathsf{row}} + \omega_{\mathsf{col}}) \right\}.$$

By virtue of (184), (187) and Lemma 5, we have

$$\mathbb{P}(\mathcal{E}_2) \ge 1 - O(e^{-c_1 x^2}) - n_1 n_2 \varepsilon$$

On the event \mathcal{E}_2 , one can obtain

$$\begin{split} \left\| \mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} \right) \right\| &\leq \left\| \mathcal{P}_{\text{off-diag}} \left(\widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top} \right) \right\| + \left\| \mathcal{P}_{\text{off-diag}} \left(\boldsymbol{E} \boldsymbol{E}^{\top} - \widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top} \right) \right\| \\ &\leq B^{2} x^{4} + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) x^{2} + \left\| \boldsymbol{E} \boldsymbol{E}^{\top} - \widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{E}}^{\top} \right\| \\ &\leq B^{2} x^{4} + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) x^{2} + \left\| \left(\boldsymbol{E} - \widetilde{\boldsymbol{E}} \right) \widetilde{\boldsymbol{E}}^{\top} \right\| + \left\| \widetilde{\boldsymbol{E}} \left(\boldsymbol{E} - \widetilde{\boldsymbol{E}} \right)^{\top} \right\| \\ &+ \left\| \left(\boldsymbol{E} - \widetilde{\boldsymbol{E}} \right) \left(\boldsymbol{E} - \widetilde{\boldsymbol{E}} \right)^{\top} \right\| \\ &\leq B^{2} x^{4} + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) x^{2} + 2 \left\| \boldsymbol{E} - \widetilde{\boldsymbol{E}} \right\| \left\| \widetilde{\boldsymbol{E}} \right\| + \left\| \boldsymbol{E} - \widetilde{\boldsymbol{E}} \right\|^{2} \\ &\lesssim B^{2} x^{4} + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) x^{2} + \frac{\omega_{\text{max}}}{n^{4}} \left(Bx + \left(\omega_{\text{row}} + \omega_{\text{col}} \right) \right) + \frac{\omega_{\text{max}}^{2}}{n^{8}} \\ &\lesssim B^{2} x^{4} + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) x^{2} + \frac{1}{2} \left(B^{2} x^{2} + \frac{\omega_{\text{max}}^{2}}{n^{8}} \right) + \frac{\omega_{\text{max}}}{n^{4}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) + \frac{\omega_{\text{max}}^{2}}{n^{8}} \\ &\lesssim B^{2} x^{4} + \omega_{\text{col}} \left(\omega_{\text{row}} + \omega_{\text{col}} \right) x^{2} \end{split}$$

for any $x \ge C_1 \sqrt{\log n}$, where the penultimate line is due to the AM-GM inequality.

Finally, we make note of a result that controls the projection of X onto the subspace spanned by \hat{U}_{\perp} (the orthogonal complement of the leading rank-*r* left singular subspace of Y).

Lemma 8 (Zhang and Xia (2018), Lemma 6). Suppose that $\mathbf{Y} = \mathbf{X} + \mathbf{E}$, where \mathbf{X} is a rank-r matrix and \mathbf{E} is the noise matrix. Let $\widehat{\mathbf{U}}$ denote the rank-r leading left singular subspace of \mathbf{Y} , and let $\widehat{\mathbf{U}}_{\perp}$ represent the orthogonal complement of $\widehat{\mathbf{U}}$. Then it holds that

$$\left\| \mathcal{P}_{\widehat{U}_{\perp}} X \right\| \leq 2 \left\| E \right\|.$$

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