Alternating direction method of multipliers

Yuxin Chen

Princeton University, Fall 2019

Outline

- Augmented Lagrangian method
- Alternating direction method of multipliers

Two-block problem

$$\begin{aligned} \text{minimize}_{\boldsymbol{x},\boldsymbol{z}} \quad & F(\boldsymbol{x},\boldsymbol{z}) := f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} = \boldsymbol{b} \end{aligned}$$

where f_1 and f_2 are both convex

- this can also be solved via Douglas-Rachford splitting
- we will introduce another paradigm for solving this problem

Augmented Lagrangian method

Dual problem

$$\begin{aligned} \text{minimize}_{\boldsymbol{x},\boldsymbol{z}} \quad & f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} = \boldsymbol{b} \end{aligned}$$

$$\Updownarrow$$

$$\text{maximize}_{\boldsymbol{\lambda}} \quad \min_{\boldsymbol{x},\boldsymbol{z}} \underbrace{f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} \rangle}_{=: \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\lambda}) \text{ (Lagrangian)}}$$

$$\Updownarrow$$

$$\text{maximize}_{\boldsymbol{\lambda}} \quad -f_1^*(-\boldsymbol{A}^\top \boldsymbol{\lambda}) - f_2^*(-\boldsymbol{B}^\top \boldsymbol{\lambda}) - \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle$$

$$\Updownarrow$$

$$\text{minimize}_{\boldsymbol{\lambda}} \quad f_1^*(-\boldsymbol{A}^\top \boldsymbol{\lambda}) + f_2^*(-\boldsymbol{B}^\top \boldsymbol{\lambda}) + \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle$$

Augmented Lagrangian method

$$\text{minimize}_{\boldsymbol{\lambda}} \quad f_1^*(-\boldsymbol{A}^\top \boldsymbol{\lambda}) + f_2^*(-\boldsymbol{B}^\top \boldsymbol{\lambda}) + \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle$$

The proximal point method for solving this dual problem:

$$\boldsymbol{\lambda}^{t+1} = \arg\min_{\boldsymbol{\lambda}} \left\{ f_1^*(-\boldsymbol{A}^\top \boldsymbol{\lambda}) + f_2^*(-\boldsymbol{B}^\top \boldsymbol{\lambda}) + \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle + \frac{1}{2\rho} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^t\|_2^2 \right\}$$

As it turns out, this is equivalent to the augmented Lagrangian method (or the method of multipliers)

$$\begin{aligned} \left(\boldsymbol{x}^{t+1}, \boldsymbol{z}^{t+1}\right) &= \arg\min_{\boldsymbol{x}, \boldsymbol{z}} \left\{ f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) + \frac{\rho}{2} \left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} + \frac{1}{\rho} \boldsymbol{\lambda}^t \right\|_2^2 \right\} \\ \boldsymbol{\lambda}^{t+1} &= \boldsymbol{\lambda}^t + \rho\left(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b}\right) \end{aligned} \tag{10.1}$$

Justification of (10.1)

$$\boldsymbol{\lambda}^{t+1} = \arg\min_{\boldsymbol{\lambda}} \left\{ f_1^*(-\boldsymbol{A}^\top \boldsymbol{\lambda}) + f_2^*(-\boldsymbol{B}^\top \boldsymbol{\lambda}) + \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle + \frac{1}{2\rho} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^t\|_2^2 \right\}$$

$\Updownarrow$ optimality condition

$$\boldsymbol{0} \in -\boldsymbol{A} \partial f_1^*(-\boldsymbol{A}^\top \boldsymbol{\lambda}^{t+1}) - \boldsymbol{B} \partial f_2^*(-\boldsymbol{B}^\top \boldsymbol{\lambda}^{t+1}) + \boldsymbol{b} + \frac{1}{\rho}\big(\boldsymbol{\lambda}^{t+1} - \boldsymbol{\lambda}^t\big)$$

$$\Updownarrow$$

$$\boldsymbol{\lambda}^{t+1} = \boldsymbol{\lambda}^t + \rho\big(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b}\big)$$

where (check: use the conjugate subgradient theorem)

$$\boldsymbol{x}^{t+1} := \arg\min_{\boldsymbol{x}} \left\{ \langle \boldsymbol{A}^\top \boldsymbol{\lambda}^{t+1}, \boldsymbol{x} \rangle + f_1(\boldsymbol{x}) \right\}$$

$$\boldsymbol{z}^{t+1} := \arg\min_{\boldsymbol{z}} \left\{ \langle \boldsymbol{B}^\top \boldsymbol{\lambda}^{t+1}, \boldsymbol{z} \rangle + f_2(\boldsymbol{z}) \right\}$$

Justification of (10.1)

$$\Updownarrow$$

$$\boldsymbol{x}^{t+1} := \arg\min_{\boldsymbol{x}} \left\{ \langle \boldsymbol{A}^\top \left[\boldsymbol{\lambda}^t + \rho \left(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b} \right) \right], \boldsymbol{x} \rangle + f_1(\boldsymbol{x}) \right\}$$

$$\boldsymbol{z}^{t+1} := \arg\min_{\boldsymbol{z}} \left\{ \langle \boldsymbol{B}^\top \left[\boldsymbol{\lambda}^t + \rho \left(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b} \right) \right], \boldsymbol{z} \rangle + f_2(\boldsymbol{z}) \right\}$$

$$\Updownarrow$$

$$\boldsymbol{0} \in \boldsymbol{A}^\top \left[\boldsymbol{\lambda}^t + \rho \left(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b} \right) \right] + \partial f_1(\boldsymbol{x}^{t+1})$$

$$\boldsymbol{0} \in \boldsymbol{B}^\top \left[\boldsymbol{\lambda}^t + \rho \left(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b} \right) \right] + \partial f_2(\boldsymbol{z}^{t+1})$$

$$\Updownarrow$$

$$\left(\boldsymbol{x}^{t+1}, \boldsymbol{z}^{t+1} \right) = \arg\min_{\boldsymbol{x}, \boldsymbol{z}} \left\{ f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) + \frac{\rho}{2} \left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} + \frac{1}{\rho} \boldsymbol{\lambda}^t \right\|_2^2 \right\}$$

Augmented Lagrangian method (ALM)

$$\left(\boldsymbol{x}^{t+1}, \boldsymbol{z}^{t+1}\right) = \arg\min_{\boldsymbol{x},\boldsymbol{z}} \left\{ f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) + \frac{\rho}{2}\left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \right\|_2^2 \right\}$$

(primal step)

$$\boldsymbol{\lambda}^{t+1} = \boldsymbol{\lambda}^t + \rho\left(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b}\right)$$

(dual step)

where $\rho > 0$ is penalty parameter

ALM aims to solve the following problem by alternating between primal and dual updates

$$\text{maximize}_{\boldsymbol{\lambda}} \ \max_{\boldsymbol{x},\boldsymbol{z}} \ \underbrace{f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) + \rho\langle \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b}, \boldsymbol{\lambda}\rangle + \frac{\rho}{2}\left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda} \right\|_2^2}_{\mathcal{L}_\rho(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\lambda}):\ \text{augmented Lagrangian}}$$

Issues of augmented Lagrangian method

$$\left(\boldsymbol{x}^{t+1}, \boldsymbol{z}^{t+1}\right) = \arg\min_{\boldsymbol{x},\boldsymbol{z}} \left\{ f_1(\boldsymbol{x}) + f_2(\boldsymbol{z}) + \frac{\rho}{2}\left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \right\|_2^2 \right\}$$

- the primal update step is often expensive — as expensive as solving the original problem
- minimization of $\boldsymbol{x}$ and $\boldsymbol{z}$ cannot be carried out separately

Alternating direction method of multipliers

Alternating direction method of multipliers

Rather than computing exact primal estimate for ALM, we might minimize $\boldsymbol{x}$ and $\boldsymbol{z}$ sequentially via alternating minimization

$$\boldsymbol{x}^{t+1} = \arg\min_{\boldsymbol{x}} \left\{ f_1(\boldsymbol{x}) + \frac{\rho}{2} \left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z}^t - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \right\|_2^2 \right\}$$

$$\boldsymbol{z}^{t+1} = \arg\min_{\boldsymbol{z}} \left\{ f_2(\boldsymbol{z}) + \frac{\rho}{2} \left\| \boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \right\|_2^2 \right\}$$

$$\boldsymbol{\lambda}^{t+1} = \boldsymbol{\lambda}^t + \rho\left(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b}\right)$$

— *called the alternating direction method of multipliers (ADMM)*

Alternating direction method of multipliers

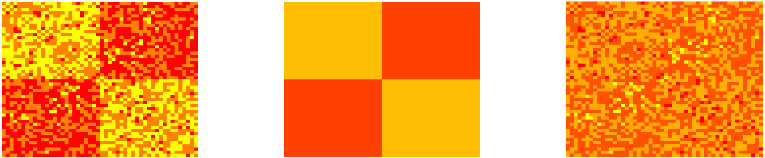
$$\boldsymbol{x}^{t+1} = \arg\min_{\boldsymbol{x}} \left\{ f_1(\boldsymbol{x}) + \frac{\rho}{2}\left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z}^t - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \right\|_2^2 \right\}$$

$$\boldsymbol{z}^{t+1} = \arg\min_{\boldsymbol{z}} \left\{ f_2(\boldsymbol{z}) + \frac{\rho}{2}\left\| \boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \right\|_2^2 \right\}$$

$$\boldsymbol{\lambda}^{t+1} = \boldsymbol{\lambda}^t + \rho\big(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b}\big)$$

- useful if updating $\boldsymbol{x}^t$ and updating $\boldsymbol{z}^t$ are both inexpensive
- blend the benefits of dual decomposition and augmented Lagrangian method
- the roles of $\boldsymbol{x}$ and $\boldsymbol{z}$ are *almost* symmetric, but not quite

Example: robust PCA



$$\boldsymbol{M} = \underbrace{\boldsymbol{L}}_{\text{low-rank}} + \underbrace{\boldsymbol{S}}_{\text{sparse}}$$

Suppose we observe $\boldsymbol{M}$, which is the superposition of a low-rank component $\boldsymbol{L}$ and sparse outliers $\boldsymbol{S}$

Can we hope to disentangle $\boldsymbol{L}$ and $\boldsymbol{S}$?

Example: robust PCA

One way to solve it is via convex programming (Candes et al. '08)

$$\begin{aligned} \text{minimize}_{\boldsymbol{L},\boldsymbol{S}} \quad & \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 \\ \text{s.t.} \quad & \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{M} \end{aligned} \tag{10.2}$$

where $\|\boldsymbol{L}\|_* := \sum_{i=1}^n \sigma_i(\boldsymbol{L})$ is the nuclear norm, and
$\|\boldsymbol{S}\|_1 := \sum_{i,j} |S_{i,j}|$ is the entrywise ℓ_1 norm

Example: robust PCA

ADMM for solving (10.2):

$$
\begin{aligned}
\boldsymbol{L}^{t+1} &= \arg\min_{\boldsymbol{L}} \left\{ \|\boldsymbol{L}\|_* + \frac{\rho}{2}\Big\|\boldsymbol{L} + \boldsymbol{S}^t - \boldsymbol{M} + \frac{1}{\rho}\boldsymbol{\Lambda}^t\Big\|_{\mathrm{F}}^2 \right\} \\
\boldsymbol{S}^{t+1} &= \arg\min_{\boldsymbol{S}} \left\{ \lambda\|\boldsymbol{S}\|_1 + \frac{\rho}{2}\Big\|\boldsymbol{L}^{t+1} + \boldsymbol{S} - \boldsymbol{M} + \frac{1}{\rho}\boldsymbol{\Lambda}^t\Big\|_{\mathrm{F}}^2 \right\} \\
\boldsymbol{\Lambda}^{t+1} &= \boldsymbol{\Lambda}^t + \rho\big(\boldsymbol{L}^{t+1} + \boldsymbol{S}^{t+1} - \boldsymbol{M}\big)
\end{aligned}
$$

Example: robust PCA

This is equivalent to

$$\boldsymbol{L}^{t+1} = \mathsf{SVT}_{\rho^{-1}}\Big(\boldsymbol{M} - \boldsymbol{S}^t - \frac{1}{\rho}\boldsymbol{\Lambda}^t\Big) \qquad \text{(singular value thresholding)}$$

$$\boldsymbol{S}^{t+1} = \mathsf{ST}_{\lambda\rho^{-1}}\Big(\boldsymbol{M} - \boldsymbol{L}^{t+1} - \frac{1}{\rho}\boldsymbol{\Lambda}^t\Big) \qquad \text{(soft thresholding)}$$

$$\boldsymbol{\Lambda}^{t+1} = \boldsymbol{\Lambda}^t + \rho\left(\boldsymbol{L}^{t+1} + \boldsymbol{S}^{t+1} - \boldsymbol{M}\right)$$

where for any $\boldsymbol{X}$ with SVD $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ ($\boldsymbol{\Sigma} = \mathrm{diag}(\{\sigma_i\})$), one has

$$\mathsf{SVT}_\tau(\boldsymbol{X}) = \boldsymbol{U}\mathrm{diag}\big(\{(\sigma_i - \tau)_+\}\big)\boldsymbol{V}^\top$$

and

$$\big(\mathsf{ST}_\tau(\boldsymbol{X})\big)_{i,j} = \begin{cases} X_{i,j} - \tau, & \text{if } X_{i,j} > \tau \\ 0, & \text{if } |X_{i,j}| \le \tau \\ X_{i,j} + \tau, & \text{if } X_{i,j} < -\tau \end{cases}$$

Example: graphical lasso

When learning a sparse Gaussian graphical model, one resorts to:

$$\begin{aligned} \text{minimize}_{\boldsymbol{\Theta}} \quad & \underbrace{-\log\det \boldsymbol{\Theta} + \langle \boldsymbol{\Theta}, \boldsymbol{S} \rangle}_{\text{negative log-likelihood of Gaussian graphical model}} \quad + \quad \underbrace{\lambda \|\boldsymbol{\Theta}\|_1}_{\text{encourage sparsity}} \\ \text{s.t.} \quad & \boldsymbol{\Theta} \succeq \boldsymbol{0} \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \text{minimize}_{\boldsymbol{\Theta}} \quad & -\log\det \boldsymbol{\Theta} + \langle \boldsymbol{\Theta}, \boldsymbol{S} \rangle + \mathbb{I}_{\mathbb{S}_+}(\boldsymbol{\Theta}) + \lambda \|\boldsymbol{\Psi}\|_1 \\ \text{s.t.} \quad & \boldsymbol{\Theta} = \boldsymbol{\Psi} \end{aligned} \tag{10.3}$$

where $\mathbb{S}_+ := \{\boldsymbol{X} \mid \boldsymbol{X} \succeq \boldsymbol{0}\}$

Example: graphical lasso

ADMM for solving (10.3):

$$\begin{aligned}
\boldsymbol{\Theta}^{t+1} &= \arg\min_{\boldsymbol{\Theta}\succeq\mathbf{0}} \left\{ -\log\det\boldsymbol{\Theta} + \frac{\rho}{2}\left\|\boldsymbol{\Theta} - \boldsymbol{\Psi}^t + \frac{1}{\rho}\boldsymbol{\Lambda}^t + \frac{1}{\rho}\boldsymbol{S}\right\|_{\mathrm{F}}^2 \right\} \\
\boldsymbol{\Psi}^{t+1} &= \arg\min_{\boldsymbol{\Psi}} \left\{ \lambda\|\boldsymbol{\Psi}\|_1 + \frac{\rho}{2}\left\|\boldsymbol{\Theta}^{t+1} - \boldsymbol{\Psi} + \frac{1}{\rho}\boldsymbol{\Lambda}^t\right\|_{\mathrm{F}}^2 \right\} \\
\boldsymbol{\Lambda}^{t+1} &= \boldsymbol{\Lambda}^t + \rho\left(\boldsymbol{\Theta}^{t+1} - \boldsymbol{\Psi}^{t+1}\right)
\end{aligned}$$

Example: graphical lasso

This is equivalent to

$$\boldsymbol{\Theta}^{t+1} = \mathcal{F}_\rho\Big(\boldsymbol{\Psi}^t - \frac{1}{\rho}\boldsymbol{\Lambda}^t - \frac{1}{\rho}\boldsymbol{S}\Big)$$

$$\boldsymbol{\Psi}^{t+1} = \mathsf{ST}_{\lambda\rho^{-1}}\Big(\boldsymbol{\Theta}^{t+1} + \frac{1}{\rho}\boldsymbol{\Lambda}^t\Big) \quad \text{(soft thresholding)}$$

$$\boldsymbol{\Lambda}^{t+1} = \boldsymbol{\Lambda}^t + \rho\left(\boldsymbol{\Theta}^{t+1} - \boldsymbol{\Psi}^{t+1}\right)$$

where for $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top \succeq \boldsymbol{0}$ with $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, one has

$\mathcal{F}_\rho(\boldsymbol{X}) := \frac{1}{2}\boldsymbol{U}\mathrm{diag}\big(\{\lambda_i + \sqrt{\lambda_i^2 + \frac{4}{\rho}}\}\big)\boldsymbol{U}^\top$

Example: consensus optimization

Consider solving the following minimization problem

$$\text{minimize}_{\boldsymbol{x}} \quad \sum_{i=1}^{N} f_i(\boldsymbol{x})$$

$$\Updownarrow$$

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{N} f_i(\boldsymbol{x}_i) \quad \color{blue}{\text{(block separable)}} \\ \text{s.t.} \quad & \boldsymbol{x}_i = \boldsymbol{z} \qquad 1 \le i \le N \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{N} f_i(\boldsymbol{x}_i) \\ \text{s.t.} \quad & \boldsymbol{u} := \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \vdots \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{z} \end{aligned}$$

Example: consensus optimization

ADMM for solving this problem:

$$
\begin{aligned}
\boldsymbol{u}^{t+1} &= \arg \min_{\boldsymbol{u}=[\boldsymbol{x}_i]_{1\le i\le N}} \left\{ \sum_{i=1}^{N} f_i(\boldsymbol{x}_i) + \frac{\rho}{2} \sum_{i=1}^{N} \left\| \boldsymbol{x}_i - \boldsymbol{z}^t + \frac{1}{\rho}\boldsymbol{\lambda}_i^t \right\|_2^2 \right\} \\
\boldsymbol{z}^{t+1} &= \arg \min_{\boldsymbol{z}} \left\{ \frac{\rho}{2} \sum_{i=1}^{N} \left\| \boldsymbol{x}_i^{t+1} - \boldsymbol{z} + \frac{1}{\rho}\boldsymbol{\lambda}_i^t \right\|_2^2 \right\} \\
\boldsymbol{\lambda}_i^{t+1} &= \boldsymbol{\lambda}_i^t + \rho\big(\boldsymbol{x}_i^{t+1} - \boldsymbol{z}^{t+1}\big), \qquad 1 \le i \le N
\end{aligned}
$$

Example: consensus optimization

This is equivalent to

$$\boldsymbol{x}_i^{t+1} = \arg\min_{\boldsymbol{x}_i} \left\{ f_i(\boldsymbol{x}_i) + \frac{\rho}{2} \left\| \boldsymbol{x}_i - \boldsymbol{z}^t + \frac{1}{\rho} \boldsymbol{\lambda}_i^t \right\|_2^2 \right\} \qquad 1 \leq i \leq N$$

(can be computed in parallel)

$$\boldsymbol{z}^{t+1} = \frac{1}{N} \sum_{i=1}^{N} \left(\boldsymbol{x}_i^{t+1} + \frac{1}{\rho} \boldsymbol{\lambda}_i^t \right)$$

(gather all local iterates)

$$\boldsymbol{\lambda}_i^{t+1} = \boldsymbol{\lambda}_i^t + \rho(\boldsymbol{x}_i^{t+1} - \boldsymbol{z}^{t+1}), \qquad 1 \leq i \leq N$$

(“broadcast” $\boldsymbol{z}^{t+1}$ to update all local multipliers)

ADMM is well suited for distributed optimization!

Convergence of ADMM

Theorem 10.1 (Convergence of ADMM)

Suppose f_1 *and* f_2 *are closed convex functions, and* γ *is any constant obeying* $\gamma \geq 2\|\boldsymbol{\lambda}^*\|_2$*. Then*

$$F\big(\boldsymbol{x}^{(t)}, \boldsymbol{z}^{(t)}\big) - F^{\mathsf{opt}} \leq \frac{\|\boldsymbol{z}^0 - \boldsymbol{z}^*\|^2_{\rho \boldsymbol{B}^\top \boldsymbol{B}} + \frac{(\gamma + \|\boldsymbol{\lambda}^0\|_2)^2}{\rho}}{2(t+1)} \tag{10.4a}$$

$$\|\boldsymbol{A}\boldsymbol{x}^{(t)} + \boldsymbol{B}\boldsymbol{z}^{(t)} - \boldsymbol{b}\|_2 \leq \frac{\|\boldsymbol{z}^0 - \boldsymbol{z}^*\|^2_{\rho \boldsymbol{B}^\top \boldsymbol{B}} + \frac{(\gamma + \|\boldsymbol{\lambda}^0\|_2)^2}{\rho}}{\gamma(t+1)} \tag{10.4b}$$

where $\boldsymbol{x}^{(t)} := \frac{1}{t+1}\sum_{k=1}^{t+1} \boldsymbol{x}^k$, $\boldsymbol{z}^{(t)} := \frac{1}{t+1}\sum_{k=1}^{t+1} \boldsymbol{z}^k$*, and for any* $\boldsymbol{C}$*,* $\|\boldsymbol{z}\|^2_{\boldsymbol{C}} := \boldsymbol{z}^\top \boldsymbol{C} \boldsymbol{z}$

- convergence rate: $O(1/t)$
- iteration complexity: $O(1/\varepsilon)$

Fundamental inequality

Define

$$\boldsymbol{w} := \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix}, \ \boldsymbol{w}^t := \begin{bmatrix} \boldsymbol{x}^t \\ \boldsymbol{z}^t \\ \boldsymbol{\lambda}^t \end{bmatrix}, \ \boldsymbol{G} := \begin{bmatrix} & & \boldsymbol{A}^\top \\ & & \boldsymbol{B}^\top \\ -\boldsymbol{A} & -\boldsymbol{B} & \end{bmatrix}, \ \boldsymbol{d} := \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{b} \end{bmatrix}$$

$$\boldsymbol{H} := \begin{bmatrix} \boldsymbol{0} & & \\ & \rho \boldsymbol{B}^\top \boldsymbol{B} & \\ & & \rho^{-1} \boldsymbol{I} \end{bmatrix}, \quad \|\boldsymbol{w}\|_{\boldsymbol{H}}^2 := \boldsymbol{w}^\top \boldsymbol{H} \boldsymbol{w}$$

Lemma 10.2

For any $\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda}$, one has

$$\begin{aligned} F(\boldsymbol{x}, \boldsymbol{z}) - F(\boldsymbol{x}^{t+1}, \boldsymbol{z}^{t+1}) + \langle \boldsymbol{w} - \boldsymbol{w}^{t+1}, \boldsymbol{G}\boldsymbol{w} + \boldsymbol{d} \rangle \\ \geq \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^{t+1}\|_{\boldsymbol{H}}^2 - \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^t\|_{\boldsymbol{H}}^2 \end{aligned}$$

Proof of Theorem 10.1

Set $\boldsymbol{x} = \boldsymbol{x}^*$, $\boldsymbol{z} = \boldsymbol{z}^*$, and $\boldsymbol{w} = [\boldsymbol{x}^{*\top}, \boldsymbol{z}^{*\top}, \boldsymbol{\lambda}^\top]^\top$ in Lemma 10.2 to reach

$$F(\boldsymbol{x}^*, \boldsymbol{z}^*) - F(\boldsymbol{x}^{k+1}, \boldsymbol{z}^{k+1}) + \langle \boldsymbol{w} - \boldsymbol{w}^{k+1}, \boldsymbol{G}\boldsymbol{w} + \boldsymbol{d} \rangle \geq \underbrace{\frac{\|\boldsymbol{w} - \boldsymbol{w}^{k+1}\|_{\boldsymbol{H}}^2}{2} - \frac{\|\boldsymbol{w} - \boldsymbol{w}^{k}\|_{\boldsymbol{H}}^2}{2}}_{\text{forms telescopic sum}}$$

Summing over all $k = 0, \cdots, t$ gives

$$(t+1)F(\boldsymbol{x}^*, \boldsymbol{z}^*) - \sum_{k=1}^{t+1} F(\boldsymbol{x}^k, \boldsymbol{z}^k) + \left\langle (t+1)\boldsymbol{w} - \sum_{k=1}^{t+1} \boldsymbol{w}^k, \boldsymbol{G}\boldsymbol{w} + \boldsymbol{d} \right\rangle$$
$$\geq \frac{\|\boldsymbol{w} - \boldsymbol{w}^{t+1}\|_{\boldsymbol{H}}^2 - \|\boldsymbol{w} - \boldsymbol{w}^0\|_{\boldsymbol{H}}^2}{2}$$

If we define

$$\boldsymbol{w}^{(t)} = \frac{1}{t+1}\sum_{k=1}^{t+1} \boldsymbol{w}^k,\ \boldsymbol{x}^{(t)} = \frac{1}{t+1}\sum_{k=1}^{t+1} \boldsymbol{x}^k,\ \boldsymbol{z}^{(t)} = \frac{1}{t+1}\sum_{k=1}^{t+1} \boldsymbol{z}^k, \boldsymbol{\lambda}^{(t)} = \frac{1}{t+1}\sum_{k=1}^{t+1} \boldsymbol{\lambda}^k,$$

then from convexity of F we have

$$F(\boldsymbol{x}^{(t)}, \boldsymbol{z}^{(t)}) - \underbrace{F(\boldsymbol{x}^*, \boldsymbol{z}^*)}_{=F^{\text{opt}}} + \langle \boldsymbol{w}^{(t)} - \boldsymbol{w}, \boldsymbol{G}\boldsymbol{w} + \boldsymbol{d} \rangle \leq \frac{1}{2(t+1)} \|\boldsymbol{w} - \boldsymbol{w}^0\|_{\boldsymbol{H}}^2$$

Proof of Theorem 10.1

Further, we claim that

$$\left\langle \boldsymbol{w}^{(t)} - \boldsymbol{w}, \boldsymbol{G}\boldsymbol{w} + \boldsymbol{d} \right\rangle = \left\langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x}^{(t)} + \boldsymbol{B}\boldsymbol{z}^{(t)} - \boldsymbol{b} \right\rangle \qquad (10.5)$$

which together with preceding bounds yields

$$F(\boldsymbol{x}^{(t)}, \boldsymbol{z}^{(t)}) - F^{\mathsf{opt}} + \left\langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x}^{(t)} + \boldsymbol{B}\boldsymbol{z}^{(t)} - \boldsymbol{b} \right\rangle \leq \frac{1}{2(t+1)} \|\boldsymbol{w} - \boldsymbol{w}^0\|_{\boldsymbol{H}}^2$$

$$= \frac{1}{2(t+1)} \left\{ \|\boldsymbol{z} - \boldsymbol{z}^0\|_{\rho \boldsymbol{B}^\top \boldsymbol{B}}^2 + \frac{1}{\rho} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^0\|_2^2 \right\}$$

Notably, this holds for any $\boldsymbol{\lambda}$

Taking maximum of both sides over $\{\boldsymbol{\lambda} \mid \|\boldsymbol{\lambda}\|_2 \leq \gamma\}$ yields

$$F(\boldsymbol{x}^{(t)}, \boldsymbol{z}^{(t)}) - F^{\mathsf{opt}} + \gamma \|\boldsymbol{A}\boldsymbol{x}^{(t)} + \boldsymbol{B}\boldsymbol{z}^{(t)} - \boldsymbol{b}\|_2$$

$$\leq \frac{\left\{ \|\boldsymbol{z} - \boldsymbol{z}^0\|_{\rho \boldsymbol{B}^\top \boldsymbol{B}}^2 + \frac{\left(\gamma + \|\boldsymbol{\lambda}^0\|_2\right)^2}{\rho} \right\}}{2(t+1)} \qquad (10.6)$$

which immediately establishes (10.4a)

Proof of Theorem 10.1 (cont.)

Caution needs to be exercised since, in general, (10.6) does not establish (10.4b), since $F(\boldsymbol{x}^{(t)}, \boldsymbol{z}^{(t)}) - F^{\mathsf{opt}}$ may be negative (as $(\boldsymbol{x}^{(t)}, \boldsymbol{z}^{(t)})$ is not guaranteed to be feasible)

Fortunately, if $\gamma \geq 2\|\boldsymbol{\lambda}^*\|_2$, then standard results (e.g. Theorem 3.60 in Beck '18) reveal that $F(\boldsymbol{x}^{(t)}, \boldsymbol{z}^{(t)}) - F^{\mathsf{opt}}$ will not be "too negative", thus completing proof

Proof of Theorem 10.1

Finally, we prove (10.5). Observe that

$$\left\langle \boldsymbol{w}^{(t)} - \boldsymbol{w}, \boldsymbol{G}\boldsymbol{w} + \boldsymbol{d} \right\rangle = \underbrace{\left\langle \boldsymbol{w}^{(t)} - \boldsymbol{w}, \boldsymbol{G}(\boldsymbol{w} - \boldsymbol{w}^{(t)}) \right\rangle}_{=0 \text{ since } \boldsymbol{G} \text{ is skew-symmetric}} + \left\langle \boldsymbol{w}^{(t)} - \boldsymbol{w}, \boldsymbol{G}\boldsymbol{w}^{(t)} + \boldsymbol{d} \right\rangle$$

$$= \left\langle \boldsymbol{w}^{(t)} - \boldsymbol{w}, \boldsymbol{G}\boldsymbol{w}^{(t)} + \boldsymbol{d} \right\rangle \tag{10.7}$$

To further simplify this inner product, we use $\boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{B}\boldsymbol{z}^* = \boldsymbol{b}$ to obtain

$$\begin{aligned} \left\langle \boldsymbol{w}^{(t)} - \boldsymbol{w}, \boldsymbol{G}\boldsymbol{w}^{(t)} + \boldsymbol{d} \right\rangle &= \left\langle \boldsymbol{x}^{(t)} - \boldsymbol{x}^*, \boldsymbol{A}^\top \boldsymbol{\lambda}^{(t)} \right\rangle + \left\langle \boldsymbol{z}^{(t)} - \boldsymbol{z}^*, \boldsymbol{B}^\top \boldsymbol{\lambda}^{(t)} \right\rangle \\ &\quad + \left\langle \boldsymbol{\lambda}^{(t)} - \boldsymbol{\lambda}, -\boldsymbol{A}\boldsymbol{x}^{(t)} - \boldsymbol{B}\boldsymbol{z}^{(t)} + \boldsymbol{b} \right\rangle \\ &= \left\langle -\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{B}\boldsymbol{z}^* + \boldsymbol{b}, \boldsymbol{\lambda}^{(t)} \right\rangle + \left\langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x}^{(t)} + \boldsymbol{B}\boldsymbol{z}^{(t)} - \boldsymbol{b} \right\rangle \\ &= \left\langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x}^{(t)} + \boldsymbol{B}\boldsymbol{z}^{(t)} - \boldsymbol{b} \right\rangle \end{aligned}$$

Proof of Lemma 10.2

To begin with, ADMM update rule requires

$$-\rho \boldsymbol{A}^\top \Big(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \Big) \in \partial f_1(\boldsymbol{x}^{t+1})$$
$$-\rho \boldsymbol{B}^\top \Big(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \Big) \in \partial f_2(\boldsymbol{z}^{t+1})$$

Therefore, for any $\boldsymbol{x}, \boldsymbol{z}$,

$$f_1(\boldsymbol{x}) - f_1(\boldsymbol{x}^{t+1}) + \Big\langle \rho \boldsymbol{A}^\top \Big(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \Big), \boldsymbol{x} - \boldsymbol{x}^{t+1} \Big\rangle \geq 0$$
$$f_2(\boldsymbol{z}) - f_2(\boldsymbol{z}^{t+1}) + \Big\langle \rho \boldsymbol{B}^\top \Big(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b} + \frac{1}{\rho}\boldsymbol{\lambda}^t \Big), \boldsymbol{z} - \boldsymbol{z}^{t+1} \Big\rangle \geq 0$$

Proof of Lemma 10.2 (cont.)

Using $\boldsymbol{\lambda}^{t+1} = \boldsymbol{\lambda}^t + \rho(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b})$, setting $\tilde{\boldsymbol{\lambda}}^t := \boldsymbol{\lambda}^t + \rho(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^t - \boldsymbol{b})$, and adding above two inequalities give

$$F(\boldsymbol{x},\boldsymbol{z}) - F(\boldsymbol{x}^{t+1},\boldsymbol{z}^{t+1}) + \left\langle \begin{bmatrix} \boldsymbol{x}-\boldsymbol{x}^{t+1} \\ \boldsymbol{z}-\boldsymbol{z}^{t+1} \\ \boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}}^t \end{bmatrix}, \begin{bmatrix} \boldsymbol{A}^\top \tilde{\boldsymbol{\lambda}}^t \\ \boldsymbol{B}^\top \tilde{\boldsymbol{\lambda}}^t \\ -\boldsymbol{A}\boldsymbol{x}^{t+1} - \boldsymbol{B}\boldsymbol{z}^{t+1} + \boldsymbol{b} \end{bmatrix} - \begin{bmatrix} \boldsymbol{0} \\ \rho \boldsymbol{B}^\top \boldsymbol{B}(\boldsymbol{z}^t - \boldsymbol{z}^{t+1}) \\ \frac{1}{\rho}(\boldsymbol{\lambda}^t - \boldsymbol{\lambda}^{t+1}) \end{bmatrix} \right\rangle \geq 0 \tag{10.8}$$

Next, we'd like to simplify above inner product. Let $\boldsymbol{C} := \rho \boldsymbol{B}^\top \boldsymbol{B}$, then

$$(\boldsymbol{z}-\boldsymbol{z}^{t+1})^\top \boldsymbol{C}(\boldsymbol{z}^t - \boldsymbol{z}^{t+1}) = \frac{1}{2}\|\boldsymbol{z}-\boldsymbol{z}^{t+1}\|_{\boldsymbol{C}}^2 - \frac{1}{2}\|\boldsymbol{z}-\boldsymbol{z}^t\|_{\boldsymbol{C}}^2 + \frac{1}{2}\|\boldsymbol{z}^t-\boldsymbol{z}^{t+1}\|_{\boldsymbol{C}}^2$$

Proof of Lemma 10.2 (cont.)

Also,

$$
\begin{aligned}
&2(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{t+1})^\top (\boldsymbol{\lambda}^t - \boldsymbol{\lambda}^{t+1}) \\
&= \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{t+1}\|_2^2 - \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^t\|_2^2 + \|\tilde{\boldsymbol{\lambda}}^t - \boldsymbol{\lambda}^t\|_2^2 - \|\tilde{\boldsymbol{\lambda}}^t - \boldsymbol{\lambda}^{t+1}\|_2^2 \\
&= \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{t+1}\|_2^2 - \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^t\|_2^2 + \rho^2 \|\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^t - \boldsymbol{b}\|_2^2 \\
&\quad - \|\boldsymbol{\lambda}^t + \rho(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^t - \boldsymbol{b}) - \boldsymbol{\lambda}^t - \rho(\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^{t+1} - \boldsymbol{b})\|_2^2 \\
&= \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{t+1}\|_2^2 - \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^t\|_2^2 + \rho^2 \|\boldsymbol{A}\boldsymbol{x}^{t+1} + \boldsymbol{B}\boldsymbol{z}^t - \boldsymbol{b}\|_2^2 \\
&\quad - \rho^2 \|\boldsymbol{B}(\boldsymbol{z}^t - \boldsymbol{z}^{t+1})\|_2^2
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&2(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{t+1})^\top (\boldsymbol{\lambda}^t - \boldsymbol{\lambda}^{t+1}) \\
&\geq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{t+1}\|_2^2 - \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^t\|_2^2 - \rho^2 \|\boldsymbol{B}(\boldsymbol{z}^t - \boldsymbol{z}^{t+1})\|_2^2
\end{aligned}
$$

Proof of Lemma 10.2 (cont.)

Combining above results gives

$$\left\langle \begin{bmatrix} \boldsymbol{x} - \boldsymbol{x}^{t+1} \\ \boldsymbol{z} - \boldsymbol{z}^{t+1} \\ \boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}^t \end{bmatrix}, \begin{bmatrix} \boldsymbol{0} \\ \rho \boldsymbol{B}^\top \boldsymbol{B}(\boldsymbol{z}^t - \boldsymbol{z}^{t+1}) \\ \frac{1}{\rho}(\boldsymbol{\lambda}^t - \boldsymbol{\lambda}^{t+1}) \end{bmatrix} \right\rangle$$

$$\geq \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^{t+1}\|_{\boldsymbol{H}}^2 - \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^t\|_{\boldsymbol{H}}^2 + \frac{1}{2}\|\boldsymbol{z}^t - \boldsymbol{z}^{t+1}\|_{\boldsymbol{C}}^2 - \frac{\rho}{2}\|\boldsymbol{B}(\boldsymbol{z}^t - \boldsymbol{z}^{t+1})\|_2^2$$

$$= \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^{t+1}\|_{\boldsymbol{H}}^2 - \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^t\|_{\boldsymbol{H}}^2$$

This together with (10.8) yields

$$F(\boldsymbol{x}, \boldsymbol{z}) - F(\boldsymbol{x}^{t+1}, \boldsymbol{z}^{t+1}) + \langle \boldsymbol{w} - \boldsymbol{w}^{t+1}, \boldsymbol{G}\boldsymbol{w}^{t+1} + \boldsymbol{d} \rangle$$
$$\geq \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^{t+1}\|_{\boldsymbol{H}}^2 - \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{w}^t\|_{\boldsymbol{H}}^2$$

Since $\boldsymbol{G}$ is skew-symmetric, repeating prior argument in (10.7) gives

$$\langle \boldsymbol{w} - \boldsymbol{w}^{t+1}, \boldsymbol{G}\boldsymbol{w}^{t+1} + \boldsymbol{d} \rangle = \langle \boldsymbol{w} - \boldsymbol{w}^{t+1}, \boldsymbol{G}\boldsymbol{w} + \boldsymbol{d} \rangle$$

This immediately completes proof

Convergence of ADMM in practice

- ADMM is slow to converge to high accuracy
- ADMM often converges to modest accuracy within a few tens of iterations, which is sufficient for many large-scale applications

Beyond two-block models

Convergence is not guaranteed when there are 3 or more blocks

- e.g. consider solving

$$x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + x_3 \boldsymbol{a}_3 = \boldsymbol{0}$$

where

$$[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

3-block ADMM is divergent for solving this problem (Chen et al. '16)

Reference

[1] "*Distributed optimization and statistical learning via the alternating direction method of multipliers*," S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein, *Foundations and Trends in Machine learning*, 2011.

[2] "*A First Course in Convex Optimization Theory*," E. Ryu, W. Yin.

[3] "*First-order methods in optimization*," A. Beck, Vol. 25, SIAM, 2017.

[4] "*Mathematical optimization, MATH301 lecture notes*," E. Candes, Stanford.

[5] "*Optimization methods for large-scale systems, EE236C lecture notes*," L. Vandenberghe, UCLA.

[6] "*Large scale optimization for machine learning, ISE633 lecture notes*," M. Razaviyayn, USC.

[7] "*Modern big data optimization, IE487/587 lecture notes*," M. Hong, ISU.

Reference

[8] "*Convex optimization and algorithms*," D. Bertsekas, 2015.

[9] "*Proximal algorithms*," N. Parikh and S. Boyd, *Foundations and Trends in Optimization*, 2013.

[10] "*The direct extension of ADMM for multi-block convex minimization problems is not necessarily convergent*," C. Chen, B. He, Y. Ye, X. Yuan, *Mathematical Programming*, 2016.

[11] "*Robust principal component analysis?*," E. Candes, X. Li, Y. Ma, J. Wright, *Journal of the ACM*, 2011.

[12] "*Sparse inverse covariance estimation with the graphical lasso*," J. Friedman, T. Hastie, and R. Tibshirani, *Biostatistics*, 2008.